

The Construction of Possible Worlds

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Abstract

This paper presents the *Logic for Tense and Modality* (TM), which completely axiomatize the class of non-deterministic dynamical systems, providing resources for reasoning about past and future contingency. By taking *world states* to be total configuration states for a system, the *task relation* encodes the possible transitions between world states over a *duration*. Given a model of the bimodal language, sentences are evaluated at a *possible world* and *time* where each possible world is a function from durations to world states constrained by the task relation. By contrast with two-dimensional semantic theories which take both times and worlds to be primitive, the perpetuity principle that what is sometimes the case is possible is a theorem in TM. The paper begins by reviewing the origins of tense and modal logic in §2 to motivate the construction of possible worlds in §3 and resulting theory of tense and modality in §4. The *Appendix* provides supporting formal results in §5. Section §1 begins by characterizing tense and modality by way of the perpetuity principles which §5 derives as theorems of TM.

Keywords: Dynamical Systems, Tense, Modality, Bimodal Logic, Task Semantics

1 Introduction

Intensional semantic theories often conceive of possible worlds as complete histories of everything. Insofar as there are *temporary sentences* which are true at some times and false at others in the same possible world, the truth-value of a temporary sentence is not fixed by a possible world considered on its own. For instance, suppose that in a world w , the sentence ‘Cary is reading’ was true this morning but false in the afternoon. Merely specifying the world of evaluation w does not determine the truth-value of a temporary sentence, at least insofar as possible worlds are taken to be temporally extended histories rather than instantaneous world states.

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One response denies that there are temporary sentences, assuming instead that every sentence, perhaps implicitly, includes a reference to some time or other. An *eternalist* of this kind takes the sentence ‘Cary is reading’ to be incomplete and so to be replaced by the *permanent sentence* ‘Cary is reading at t ’ where t is a time.¹ However, including singular terms which refer to *times* makes an ontology of times a part of the topic of conversation. Although some sentences may be about certain times either imagined, fictional, experienced, theorized about, or otherwise discussed, the sentence ‘Cary is reading’ does not concern any time whatsoever, but rather has Cary and her engagement reading as the entire subject-matter. Moreover, since a permanent sentence that is true in w is true at any time in w , a permanent sentence φ is equivalent to the result of embedding φ under arbitrary tense operators, obviating the need to include tense operators in the language.² However, the truth-condition for the sentence ‘Cary is reading’ differs substantially from the result of embedding this sentence under tense operators. For instance, it might be true that Cary is reading, and false that she always has been reading, or is always going to be reading. Instead of excluding tense operators and temporary sentences such as ‘Cary is reading’ from the language, this paper provides a truth-conditional semantics which respects the natural morphology of tensed and modal claims, describing their interactions in a bimodal language without positing unspoken references to an ontology of times.

Rather than including singular terms for times in the logical form of the sentences of a language, the semantics pioneered by Montague [1] and Kaplan [2] evaluates sentences at world-time pairs. Assigning sentences to sets of world-time pairs makes times endogenous to the interpretation of the language.³ This strategy provides a two-dimensional semantics for $\Box$ and $\Box$ which read ‘It has always been the case that’ and ‘It is always going to be the case that’, respectively. By evaluating sentences at world-time pairs, it is easy to distinguish the truth-condition for ‘Cary is reading’ from the result of embedding this sentence under tense operators which shift the time of evaluation while holding the world of evaluation fixed. Whereas ‘Cary is reading’ is assigned to the set of all world-time pairs in which Cary is reading in that world at that time, ‘Cary always was reading’ is assigned to the set of all world-time pairs where she is reading at all earlier times in that world.

Although two-dimensional semantics admits distinct truth-conditions for sentences of the form φ , $\Box\varphi$, $\Box\varphi$, $\Diamond\varphi$, $\Diamond\varphi$ and sentences with iterated temporal operators, taking possible worlds to be structureless points within a model while nevertheless representing temporally extended histories comes at both an intuitive and theoretical cost. Letting $\Diamond\varphi := \neg\Box\neg\varphi$ and $\Diamond\varphi := \neg\Box\neg\varphi$ express ‘It has been the case that φ ’ and ‘It is going to be the case that φ ’, I will take $\Delta\varphi := \Box\varphi \wedge \varphi \wedge \Box\varphi$ and $\nabla\varphi := \Diamond\varphi \vee \varphi \vee \Diamond\varphi$ to read ‘It is always the case that φ ’ and ‘It is sometimes the case that φ ’ where $\Box$ and $\Diamond$ are the metaphysical modals which shift the world of evaluation while holding the time fixed. A semantics of this kind makes the following *perpetuity principles* invalid:

¹Atomic permanent sentences could be regimented by ‘ $R_t(c)$ ’ or ‘ $R(c, t)$ ’. What matters is that the time t is included in the logical form rather than as a parameter at which the sentence is evaluated.

²There are pathological cases if time is bounded, though these exceptions are beside the present point.

³Alternatively, sentences could be assigned to characteristic functions from world-time pairs to truth-values, or else functions from times (perhaps together with other contextual parameters) to functions from worlds to truth-values, all of which take times to be endogenous to the interpretation of the language.

$$\mathbf{P1} \quad \Box\varphi \rightarrow \Delta\varphi.$$

$$\mathbf{P2} \quad \nabla\varphi \rightarrow \Diamond\varphi.$$

So long as sentence letters may be assigned to any set of world-time pairs where there is more than one time, φ may be true in every world w' at a time x without also being true in a world w at every time x' . Insofar as $\Box\varphi$ is true in a world w at time x just in case φ is true in u at time x for every world u , it follows that **P1** is invalid where **P2** is also invalid on account of being logically equivalent. However, it is natural to assume that whatever is metaphysically necessary is always the case, or equivalently, whatever is sometimes the case is metaphysically possible.

To bring out the metaphysical reading of the modal operators $\Box$ and $\Diamond$ that I will assume, consider Alvin who recently turned ten years old $\Diamond\text{Ten}$. Although it follows that there is a time where Alvin turned ten ∇Ten , one might deny that it is *now* possible for Alvin to turn ten $\neg\Diamond\text{Ten}$, and so **P2** is false. It may then be taken to be an advantage to make **P1** and **P2** invalid to accommodate such examples. This example highlights that $\Box$ and $\Diamond$ cannot be assimilated to the natural language expressions ‘necessarily’ and ‘possibly’ which quantify over a contextually restricted set of possible worlds rather than all objective possible worlds. For instance, it is often natural to restrict consideration to some relevant subset of possible worlds that overlap with the past leading up to the present. By contrast, the metaphysical modals quantify over all possible worlds, including those that bear no relation to the actual past from our present perspective. In the case of Alvin turning ten years old, we may consider possible worlds just like the world imagined to be actual, in which Alvin turned ten years old some time ago but where events are shifted forward such that he is just now turning ten years old. Insofar as the metaphysical modal operators quantify over all such possible worlds, it follows immediately that it *is* possible for Alvin to turn ten years old, though not in any possible world that shares our actual past.

I will take metaphysical modality to be the strongest objective modality.⁴ Letting ‘ $\varphi \equiv \psi$ ’ read ‘For it to be the case that φ just is for it to be the case that ψ ’ where ‘ $\vec{\chi}_{(\psi/\varphi)}$ ’ is the result of freely substituting ψ for φ among the arguments $\vec{\chi}$, an n -place operator $\mathcal{Q}$ is *objective* only if all instances of the following schema hold:

$$\text{Transparency: } (\varphi \equiv \psi) \rightarrow [\mathcal{Q}(\vec{\chi}) \equiv \mathcal{Q}(\vec{\chi}_{(\psi/\varphi)})].^5$$

Letting $\Box$ read ‘It is nomologically necessary that’, I will assume that $\Box$ is also an objective modal operator, but strictly weaker than the metaphysical modals. Rather than quantifying over all possible worlds, $\Box\varphi$ is true in a world w at time x just in case φ is true in u at x for every world u which conforms to the laws of nature in w at x . Whereas $\Box\varphi \rightarrow \Box\varphi$ is valid, $\Box\varphi \rightarrow \Diamond\varphi$ admits counterexamples insofar as there are possible worlds that do not obey the laws of nature. In contrast to the metaphysical and nomological modals, the belief operator ‘Homer believes that’ is not transparent, and so fails to be objective. For instance, ‘Homer believes Hesperus is rising’ and ‘Homer believes that Phosphorus is rising’ may differ in truth-value despite the fact

⁴See Williamson [3–5] for discussion. The formal characterization of $\Box$ as the strongest objective normal modal operator is developed in §5.1, where such an operator is shown to be unique up to equivalence where existence is derived under a coarse-grained strengthening of propositional identity in C1.

⁵This transparency schema states a necessary condition on objectivity. See §5.1 in the *Appendix* for the quantified transparency condition relative to a logic, where the definition of $\text{Obj}(\mathcal{Q})$ builds in factivity.

that Hesperus is Phosphorus and so ‘Hesperus is rising’ and ‘Phosphorus is rising’ express the same proposition, though this was unknown to Homer.

This paper presents an intensional *task semantics* for the bimodal language $\mathcal{L}$ which includes tense and modal operators for reasoning about non-deterministic dynamical systems.⁶ Instead of taking both possible worlds and times to be primitive as assumed by two-dimensional semantic theories, §3 introduces primitive *world states*, *tasks*, and *durations* to define possible worlds as functions from durations to world states which are constrained by the task relation. By taking durations to be primitive, a *moment* (equivalently, a *time*) is only fixed by a duration relative to the choice of temporal origin in a possible world, and so modeled by a world-duration pair which I will refer to as a *world-time pair* for familiarity. By evaluating sentences at world-time pairs where the semantics for $\Box$ and $\Diamond$ quantifies over all possible worlds and the semantics for $\Box_P$ and $\Box_F$ quantifies over the past and future times respectively, I show that the perpetuity principles are valid, strengthening the logic. After presenting the sound and complete *Logic of Tense and Modality* (TM) along with its extensions, §4 provides an account of the openness of the future before drawing further connections to dynamical systems theory and adjacent logics in computer science.⁷ The formal results referred to throughout will be provided by the *Appendix* in §5. Prior to these developments, the following section will motivate the construction of possible worlds by reviewing the twentieth century history of tense and modal logic.

2 Primitive Worlds

In order to provide a flexible semantics with which to characterize the modal systems that Lewis and Langford [7] first set out in Appendix II of their 1932 textbook, Kripke [8, 9] took the models of a modal language to include a nonempty set of *possible worlds* W , an *evaluation world* w , an *accessibility relation* R , and an *interpretation* $\mathcal{I}$ assigning each sentence letter to a truth-value at each world. Adding these resources improved on Carnap’s [10, 11] semantic theories that evaluated sentences at *state-descriptions* which, for any atomic sentence of the language, include that sentence or its negation but not both. Carnap [10] specified how state-descriptions are to be understood:

A state description is a class of sentences which represents a possible specific state of affairs by giving a complete description of the universe of individuals with respect to all properties and relations designated by predicates in the system.⁸ (p. 50)

Since Carnap’s state-descriptions are entirely syntactic in their construction and so determined by the primitive symbols included in a non-modal language, Carnap does not provide a genuine model theory but rather a single structure by which to interpret a modal language. Building on Carnap’s efforts, Kripke [13] first sought to evaluate

⁶I present a hyperintensional task semantics in [6], defining world states as maximal possible states.

⁷The Lean 4 repository <https://github.com/benbrastmckie/BimodalLogic> for this paper implements the soundness and completeness results as well as a small library of derivations.

⁸In his earlier work, Carnap [12, p. 95] writes, “If a false sentence is not L-false, hence not self-contradictory, it describes a situation which [is] possible though not real,” and later that, “a system S has to do with many objects, and hence we have to consider the possible states of affairs of all the objects dealt with in S and with respect to all properties, relations, etc., dealt with in S ,” (p. 101) indicating Carnap’s concern with an objective modality that quantifies over *possible circumstances* rather than the interpretational modality that the formal details of Carnap’s theory are in a better position to support.

sentences at *complete assignments* over a *domain* D where each assignment maps the singular terms to elements in D and n -adic predicates to sets of n -tuples of elements in D where sentence letters are assigned to truth or falsity. By contrast with Carnap's state-descriptions which belong to a single fully specified structure, Kripke took the domain D included in a model to be any set whatsoever, where it was by quantifying over all such models that Kripke defined validity for the language. Given that there are no two complete assignments that agree on all elements of the language, the range of complete assignments is determined by the primitive symbols included in the language together with the domain provided by the models of the language.

Despite the language relativity of both Carnap's state-descriptions and Kripke's complete assignments modeled after them, these constructions sought to represent *possibilities* to provide semantic clauses for modal operators by quantifying over all or some possibilities. Whereas objective modality concerns possible ways for a system to be configured independent of the expressive resources included in any language, *interpretational modality* concerns consistent ways of interpreting a language, making a degree of language-relativity both inherent and appropriate. Although the language relativity of Carnap [10, 11] and Kripke's [13] early accounts presents no issue for the interpretational modals that they may be taken to have described, Kripke sought to accommodate a broader range of readings for the modal operators by decoupling the possibilities over which the modal operators quantified from the primitive symbols in the language.⁹ Beginning with the propositional fragment, Kripke [8] introduced a primitive set of *possible worlds* W which he took to be any nonempty set whatsoever, where the interpretation of the language was handled by an *interpretation function* mapping each sentence letter and possible world to a truth-value.¹⁰ By also specifying a primitive relation R over W for *relative possibility*, Kripke showed how to associate the reflexivity, symmetry, and transitivity constraints on R with the corresponding **T**, **B**, and **4** axioms which characterized the differences between the most prominent modal systems that Lewis and Langford [7] had introduced.¹¹ Although Kripke [9] went on to extend his semantics to include predicates and first-order quantifiers, it will suffice for present purposes to restrict consideration to the propositional fragment of the languages with which Carnap and Kripke were concerned.

Despite their differences, neither Carnap nor Kripke were focused on interpreting languages with tense operators or singular terms for an ontology of times.¹² Insofar as state-descriptions, complete assignments, and possible worlds are taken to determine the truth-values of temporary sentences, it is implausible to interpret these elements as temporally extended histories. Besides being irrelevant to the interpretation of

⁹Kripke [14, 15] went on to describe a metaphysical reading of the modal operators for which it is inappropriate to relativize the range of possibilities to the expressive power of a language.

¹⁰Officially, Kripke [8] took the *models* on a *model structure* $\langle W, R, w \rangle$ to be functions which take a propositional variable and world to a truth-value. I have replaced propositional variables with sentence letters and reserved the label 'model' to improve consistency with the definitions presented below.

¹¹The relational structures underlying Kripke semantics were anticipated algebraically by Jónsson and Tarski's representation theorem for Boolean algebras with operators [16, 17], which embeds every such algebra into a powerset algebra over a set with a binary relation— i.e., a frame. However, neither Tarski nor his contemporaries recognized the connection to modal semantics. Tarski reportedly told Kripke in 1962 that he could not see a link between their respective programs. See Goldblatt [18] for historical discussion.

¹²Although Kripke interprets possible worlds as *moments* in a letter discussing Prior's [19] temporal interpretation of the modal operators, he writes, "I myself was working with ordinary modal logic." For the published correspondence and accompanying discussion, see Ploug [20, p. 373].

a modal language without tense operators, taking possible worlds to be temporally extended leaves the truth-values for temporary sentences underdetermined, requiring the addition of a temporal parameter to the point of evaluation in order to specify the time in the possible world at which the sentence is to be evaluated. However, neither Carnap nor Kripke provide any indication that times are to be considered in evaluating sentences. Since the possible worlds that Kripke took to be primitive cannot be interpreted as temporally extended, they must be instantaneous.¹³

Although it is natural to conceive of possible worlds as complete configurations of a system at a moment when interpreting the sentences of a language with either tense operators or modal operators but not both, the same cannot be said for bimodal languages with both tense and modal operators. Not only do instantaneous moments fail to specify what comes before or after when considered on their own, including an ordering of moments once and for all fails to capture the range of different orderings of moments that are possible. What is needed to interpret bimodal sentences is an encoding of both the modal and temporal dimensions of the semantics.

While working to develop a semantics for tense logic, Prior [19, 21] took the first steps towards constructing the resources needed to interpret a bimodal language. Rather than restricting consideration to operators that quantify over instantaneous configurations of a system, Prior defined world histories to consist of temporally ordered configuration states. In addition to the tense operators which quantify over the past and future states in a history, Prior introduced operators that quantify over world histories to characterize the open future. The following section will review Prior's efforts, highlighting his most important insight that world histories are to be defined rather than primitive while also considering the limitations this approach faces.

2.1 World States

In *Time and Modality*, Prior [19] developed a Diodorean interpretation of $\Diamond\varphi$ at a sequence of numbers which he took to represent the future.¹⁴ Whereas the first number in the sequence represented the truth-value of φ in the present, the subsequent numbers represent the truth-value of φ at incrementally later times.

Commenting on Prior's book, Kripke observed in a letter [20, p. 370] that the Diodorean system validates $\Box\Diamond\varphi \vee \Box\Diamond\neg\varphi$ which does not belong to an S4 logic. Kripke went on to suggest a branching structure in which the past is determined but the future remains open, where models of this kind are more appropriate to the S4 logic that Prior had discussed. Crediting Kripke for this account in [21, p. 27], Prior developed a number of semantic theories and corresponding logics for tensed languages where sentences are evaluated at *world states* which model, "instantaneous total states of the world," (p. 88) rather than temporally extended histories. Intuitively,

¹³In the only example that Kripke [9] provides, the sentence 'Sherlock Holmes is bald' has a truth-value even though a time has not been specified and the name 'Sherlock Holmes' does not refer. Assuming Cary to be in the flesh and blood, 'Cary is reading' may be said to have a truth-value with considerably less controversy. In both cases, Kripke's semantics returns a truth-value despite the fact that a time has not been included in the sentence in question nor among the parameters at which that sentence is interpreted.

¹⁴Diodorus Cronus (died c. 284 BCE) was a Megarian logician who defined possibility and necessity in temporal terms, where a proposition is possible just in case it either is or will be true, and necessary just in case it is true and always will be true. Although Prior was concerned to provide a semantics and logic for a tensed language without metaphysical modal operators, he stops short of claiming that the metaphysical modals are definable in terms of the temporal operators with which he was concerned.

each world state is a complete configuration of the system under study at an instant, individuated by the intrinsic properties of that system in question. In order to interpret the tense operators, Prior included a strict *earlier-than* relation $<$ to order the world states. Situating his semantic approach in contrast to previous developments, Prior [21, p. 7-8] cites Broad's [22, p. 315] criticism of McTaggart's [23] famous argument that time is unreal, writing that the fundamental error in McTaggart's argument is his attempt to, "impose conditions appropriate to a tenseless language upon a tensed one." Prior [21, p. 12] avoids this defect by following Broad's suggestion to, "drop the temporal predicates 'past', 'present', and 'future'."¹⁵ Whereas McTaggart includes times, temporal predicates, and tense operators all in one language in order to present his arguments, Prior interprets an object language with tense operators by appealing to an extensional metalanguage in which world states are ordered by an earlier-than relation but no other temporal relations or properties are included.

Although the world states form a strict total order in one of the simplest versions of his semantics, Prior goes on to consider models which accommodate an open future by taking the world states to form a strict partial order that is connected and left-linear rather than total.¹⁶ Despite these differences, $<$ is a strict partial order on each account that Prior considered since no world state is permitted to be earlier than itself, and a world state s is earlier than r if both s is earlier than t and t is earlier than r . It is important to observe that taking world states to be strictly ordered prevents the same world state from occurring more than once in any history for that system. However, supposing that history never repeats itself is a substantive assumption which should not be built into the semantics. Moreover, there are systems which we may wish to study that admit loops in their evolution. For instance, taking the system in question to be a chessboard, there are histories for that system which include meandering end games where the same board state occurs more than once. Nothing should rule out consideration of such games of chess. Rather, this example highlights a fundamental limitation of the theoretical roles that world states play in Prior's semantics. Although it is natural to consider systems which occupy the same instantaneous configuration at different times, this is forbidden if world states are strictly ordered.

To disentangle the distinct theoretical roles which world states play in Prior's semantics, it will help to define a minimally constrained class of models that generalize on Thomason's [25] reconstruction of the Peircean and Ockhamist semantic theories that Prior [21] develops. Letting $\mathcal{L}^T := \langle \mathbb{L}, \perp, \rightarrow, \Box, \Diamond \rangle$ be a non-modal propositional language, a *strict model* of $\mathcal{L}^T$ is an ordered triple $\mathcal{P} = \langle T, <, |\cdot| \rangle$ where $\langle T, < \rangle$ is a strict partial order for a nonempty set of world states T , and $|p_i| \subseteq T$ for all sentence letters $p_i \in \mathbb{L}$.¹⁷ Although some strict models correspond to a single history for a system by also being total, strict models may accommodate distinct histories which do not intersect or only partially overlap. A *strict history* in a strict model $\langle T, <, |\cdot| \rangle$

¹⁵Instead of following Reichenbach [24] in distinguishing between the point-of-speech, point-of-reference, and point-of-event to evaluate tensed sentences at a single world state, it is by evaluating sentences with nested tense operators that the evaluation time shifts between the right number of world states.

¹⁶A *strict partial order* $\langle T, < \rangle$ is a set T equipped with an irreflexive and transitive relation $<$. Letting $x \sim y := (x < y) \vee (x = y) \vee (x > y)$ express that x and y are comparable, a strict partial order $\langle T, < \rangle$ is *total* if $s \sim t$ for any $s, t \in T$. A strict partial order $\langle T, < \rangle$ is *left-linear* just in case $s \sim t$ for any $s, t, r \in T$ where $s < r$ and $t < r$. Letting $\tilde{\sim}$ be the transitive closure of $\sim$, a frame $\langle T, < \rangle$ is *connected* just in case $x \tilde{\sim} y$ for all $x, y \in T$. A left-linear frame is connected if for any $x, y \in T$, there is some z where $z < x$ and $z < y$.

¹⁷I will assume $\neg\varphi := \varphi \rightarrow \perp$, $\varphi \vee \psi := \neg\varphi \rightarrow \psi$, $\varphi \wedge \psi := \neg(\varphi \rightarrow \neg\psi)$, and $\varphi \leftrightarrow \psi := (\varphi \rightarrow \psi) \wedge (\psi \rightarrow \varphi)$.

is any maximal total suborder $h_i = \langle T_i, <_i \rangle$ of $\langle T, < \rangle$.¹⁸ Letting $H_{\mathcal{P}}$ be the set of all strict histories of a strict model $\mathcal{P}$, we may take $H_{\mathcal{P}}^x := \{\langle T_i, <_i \rangle \in H_{\mathcal{P}} \mid x \in T_i\}$ to be the strict histories $\langle T_i, <_i \rangle$ of $\mathcal{P}$ which include the world state $x \in T_i$.

Despite being defined rather than primitive, strict histories play a similar role to possible worlds in the semantics that Montague [1] and Kaplan [2] went on to provide since each strict history corresponds to a complete temporal evolution of the system in question. In particular, consider the following semantic clauses:

$$\begin{aligned} \text{Peircean: } \mathcal{P}, x \models \Box^{\text{P}} \varphi &\text{ iff } \mathcal{P}, y \models \varphi \text{ for some } h_i \in H_{\mathcal{P}}^x \text{ and all } y \in h_i \text{ where } y <_i x. \\ \mathcal{P}, x \models \Box^{\text{F}} \varphi &\text{ iff } \mathcal{P}, y \models \varphi \text{ for some } h_i \in H_{\mathcal{P}}^x \text{ and all } y \in h_i \text{ where } x <_i y. \\ \text{Ockhamist: } \mathcal{P}, h_i, x \models \Box^{\text{O}} \varphi &\text{ iff } \mathcal{P}, h_i, y \models \varphi \text{ for all } y \in T_i \text{ where } y <_i x. \\ \mathcal{P}, h_i, x \models \Box^{\text{F}} \varphi &\text{ iff } \mathcal{P}, h_i, y \models \varphi \text{ for all } y \in T_i \text{ where } x <_i y. \end{aligned}$$

Whereas Prior [21, p. 126, 132] provided a semantics for the metric tense operators and Thomason [25] restricted attention to $\Diamond^{\text{P}}$ and $\Diamond^{\text{F}}$, I have derived the semantics for $\Box^{\text{P}}$ and $\Box^{\text{F}}$ from the semantics that Thomason provides while generalizing the account to accommodate all strict models of $\mathcal{L}^{\text{T}}$.¹⁹ The Peircean and Ockhamist semantics disagree about what the tense operators express at any world state that belongs to more than one strict history. Whereas the Peircean semantics quantifies over all past or future world states in *some* strict history that includes the world state at which the tensed sentence is evaluated, the Ockhamist semantics evaluates tensed sentences at both a strict history and world state in that history, quantifying over just the world states in that strict history. As a result, the Peircean semantics has a number of unnatural consequences. For instance, given a board state in a game of chess where there is at least one history in which Black does not make any further blunders, the sentence ‘Black is not going to blunder’ is true even though there may be other histories in which Black goes on to blunder. This is far from natural. Moreover, revising the Peircean semantics to quantify over *all* strict histories in addition to all past or future world states makes the operators $\Box^{\text{P}}$ and $\Box^{\text{F}}$ too strong and their duals too weak. For instance, in a game of chess in which there is at least one future in which the white king is in checkmate and at least one future in which the black king is in checkmate, both ‘Black is going to win’ and ‘White is going to win’ come out true.

Rather than quantifying over all past or future world states in either some or all strict histories, the Ockhamist semantics evaluates sentences at both a strict history and world state. The sentences ‘Black is going to win’ and ‘White is going to win’ cannot both be true since there is no way for both kings to be in checkmate in the same game. In addition to claiming this advantage, another prominent difference between the Ockhamist and Peircean semantics is the expressive power that they each afford. Since the Ockhamist tense operators only quantify over the world states in the strict history included in the point of evaluation, the Peircean operators may be defined in terms of the Ockhamist operators given the following *stability operator*:

¹⁸Letting a *total suborder* of $\langle T, < \rangle$ be any total order $\langle T_i, <_i \rangle$ where $T_i \subseteq T$ and $<_i$ is the restriction of $<$ to T_i , a total suborder $\langle T_i, <_i \rangle$ of $\langle T, < \rangle$ is *maximal* just in case $\langle T_j, <_j \rangle = \langle T_i, <_i \rangle$ for any total suborder $\langle T_j, <_j \rangle$ of $\langle T, < \rangle$ where $\langle T_i, <_i \rangle$ is a total suborder of $\langle T_j, <_j \rangle$.

¹⁹The Peircean semantics given above can be simplified were one to restrict consideration to strict models which are left-linear (similarly right-linear), since world states would then have a unique past (future).

$$(\Box^0) \mathcal{P}, h_i, x \models \Box^0 \varphi \text{ iff } \mathcal{P}, h_j, x \models \varphi \text{ for all } h_j \in H_{\mathcal{P}}^x.$$

By contrast to the Ockhamist tense operators, the stability operator quantifies over all intersecting strict histories that occupy the same world state at which the sentence is evaluated. Prior [21, p. 125] reads ‘ $\Box^0 \varphi$ ’ as ‘It is now unpreventable that φ ’, though uses ‘Necessarily φ ’ for convenience, claiming that whereas the Peircean’s, “rather strong ‘will be’ is simply the Ockhamist ‘necessarily will be’, the Ockhamist ‘will be’ [is] untranslatable,” (p. 130) for the Peircean.²⁰ As Prior observes, $\varphi \rightarrow \Box^0 \varphi$ is valid for non-temporal sentences. $\Box$ (similarly $\Box^0$) may be included in φ by restricting to strict models that are left-linear (right-linear). Although the Ockhamist may define the Peircean tense operators $\Box^P \varphi := \Diamond^0 \Box^0 \varphi$ and $\Box^F \varphi := \Diamond^0 \Box^0 \varphi$, the Peircean is not in a position to define the Ockhamist operators for tense, nor to provide a semantics for a stability operator since sentences are evaluated at world states alone.

Despite their differences, neither the Peircean nor Ockhamist semantics permit the same world state in T to occur more than once in a strict history, nor are there strict histories in which the same world states in T occur in different orders. Rather, $\langle T, < \rangle$ is required to be a strict partial order for any strict model $\langle T, <, |\cdot| \rangle$ of $\mathcal{L}^T$, and so if $s <_i t$ for any strict history of $\langle T, <, |\cdot| \rangle$, then $t \not<_j s$ for all strict histories of $\langle T, <, |\cdot| \rangle$. However, given Prior’s conception of world states as complete configurations, there are systems in which the same world states occur more than once or in different orders in different possible histories. As brought out above, a chess game may include the same board state more than once, or two chess games may agree in all respects with the exception of a transposition of board states which occur in a different order.²¹

Insofar as T is taken to be the set of all world states of a particular system under study, the strict histories for that system do not permit the same world states to occur more than once or in different orders, significantly compromising the range of applications for the semantics. Rather than accepting these limitations, it is natural to reject Prior’s conception of T as the set of world states where these are taken to be complete configurations of the system, taking T to be the set of *times* in accordance with their strict ordering. After all, T is intended to provide resources for articulating a semantics for temporal operators which quantify over what comes before or after each element in T . On this reading, the truth-conditions for the sentences of $\mathcal{L}^T$ specify *when* a sentence is true without indicating which world state the system happens to occupy at each time. Strict models may include strict histories in which the same world states occur more than once or in different orders for the simple reason that strict models only concern the times at which sentences are evaluated without saying anything about the world states that the system occupies at those times. As a result, there is no telling which times in T occupy the same world states, or which strict histories include the same world states in different orders. For instance, supposing the white king to be in check at multiple times in a game of chess, there is no way to distinguish which times in the truth-condition for ‘The white king is in check’ correspond to the same complete configuration of the chessboard (if any) and which times correspond to distinct configurations. Moreover, given two strict histories for a chess game which

²⁰Prior [21, p. 130] attributes these observations to J.M. Shorter in 1957 but does not provide a proof.

²¹Following Müller’s [26] suggestion, Rumberg [27] extends Prior [21] and Thomason’s [25] semantics. By virtue of working over a partial order, this extension faces the same difficulties brought out here.

agree in all respects with the exception of a short sequence of moves in which the same board states occur in different orders, the divergent sequences must include distinct times in T despite reordering the same configurations of the chessboard.

Taking T to be the set of times rather than world states avoids ruling out strict histories in which the same world states occur more than once or in different orders. Since all that is required to interpret tense operators is to specify *when* each sentence is true, the erroneous interpretation of T as a set of complete configurations of a system may be forgiven when considering non-modal languages like $\mathcal{L}^T$. Despite providing an adequate range of semantic primitives for interpreting the tensed languages with which Prior and Thomason were concerned, the same cannot be said for a bimodal language with operators for both tense and metaphysical modality. In order to evaluate a sentence φ of the bimodal language $\mathcal{L}$ at a model $\mathcal{P}$, strict history h_i , and time x , it is important that $x \in T_i$ be a valid time in the strict history $h_i = \langle T_i, <_i \rangle$. As a result, the strongest modal operator that an Ockhamist model theory can provide only quantifies over those strict histories in which the time of evaluation occurs, and so cannot quantify over the full range of strict histories included in a strict model. This is exactly what the Ockhamist stability operator $\Box^o$ achieves, though Prior is careful not to conflate this modality with metaphysical necessity since the stability modal does not quantify over all strict histories whatsoever. Since there is no Ockhamist modality that is able to quantify over all strict histories, the range of strict histories cannot represent the full range of possible evolutions for the system in question.

The source of these limitations is that Prior evaluates tensed sentences at world states alone while ordering those very world states by the earlier-than relation. A single object is made to answer two different questions: *when* a sentence is true, given by its position in the strict order, and *what* configuration the system then occupies, given by the world state itself. Since the position-marker and the configuration are one and the same, no configuration can occupy two positions, which is why the same world state cannot recur in any strict history. Rather than accepting these limitations, I will take both *world states* and *durations* to be semantic primitives in §3, separating the theoretical roles that Prior conflates by assigning the *what* to the world states and the *when* to the durations. I will also include a *task relation* in order to define possible worlds as appropriately constrained functions from durations to world states, tracing out different paths through the space of all world states in alignment with standard definitions in dynamical systems theory. In addition to providing a natural model theory in which possible worlds may occupy the same world states more than once or in different orders, taking sets of world states to provide truth-conditions for the sentence letters in the language makes times exogenous to the interpretation of the language. As I show in **C2** of the *Appendix*, both **P1** and **P2** are valid over the unrestricted class of models that I define for the bimodal language.

Instead of constructing the space of possible worlds from world states, tasks, and times, Montague [1] and Kaplan [2] separate the *when* from the *what* by taking both the set of possible worlds W and the set of times T to be primitive along with a weak total order $\leq$ for the *at least as early as* relation.²² Abstracting from their differences,

²²Little turns on whether the order is weak $\leq$ or strict $<$ following Prior. I will follow Montague and Kaplan in omitting consideration of the accessibility relation between possible worlds.

let a *two-dimensional model* $\mathcal{M}_2 = \langle W, T, \leq, |\cdot| \rangle$ include an interpretation function where $|p_i| \subseteq W \times T$ is a set of world-time pairs for each sentence letter p_i with $i \in \mathbb{N}$. However natural it may seem to extend Kripke's semantics along these lines, Montague's semantics undermines the expressive power of the modal operator included in his language while Kaplan's semantics invalidates the perpetuity principles, weakening the logic that the resulting model theory supports. The following subsections will review the shortcomings of these early bimodal semantic theories before turning to present the construction of possible worlds in §3.

2.2 Necessarily Always

Whereas Montague [1] sought to provide a formal semantics for a fragment of English by regimenting that fragment in a quantified bimodal language for which he gave a recursive semantics, I will restrict consideration to the propositional fragment of Montague's formal language. To facilitate comparison, I will include $\boxtimes$ in place of $\Box$ in the propositional language $\mathcal{L}^M = \langle \mathbb{L}, \perp, \rightarrow, \boxtimes, \boxplus, \boxminus \rangle$. By defining the well-formed sentences of $\mathcal{L}^M$ in the usual way, we may extend the interpretation provided by each two-dimensional model $\mathcal{M}_2$ to all well-formed sentences of $\mathcal{L}^M$ as follows:

- $(p_i) \mathcal{M}_2, w, x \models p_i \text{ iff } \langle w, x \rangle \in |p_i|.$
- $(\perp) \mathcal{M}_2, w, x \not\models \perp.$
- $(\rightarrow) \mathcal{M}_2, w, x \models \varphi \rightarrow \psi \text{ iff } \mathcal{M}_2, w, x \not\models \varphi \text{ or } \mathcal{M}_2, w, x \models \psi.$
- $(\boxtimes) \mathcal{M}_2, w, x \models \boxtimes\varphi \text{ iff } \mathcal{M}_2, u, y \models \varphi \text{ for all } u \in W \text{ and } y \in T.$
- $(\boxplus) \mathcal{M}_2, w, x \models \boxplus\varphi \text{ iff } \mathcal{M}_2, w, y \models \varphi \text{ for all } y \in T \text{ where } y < x.$
- $(\boxminus) \mathcal{M}_2, w, x \models \boxminus\varphi \text{ iff } \mathcal{M}_2, w, y \models \varphi \text{ for all } y \in T \text{ where } x < y.$

Postponing the controversies that beset the interpretation of the future tense operator to §4.1, I will take the semantics for both the extensional and tense operators to be uncontroversial for the time being. Whereas Montague [1] took modal claims of the form $\boxtimes\varphi$ to be read 'It is necessarily the case that φ ', the same paper in the edited collection included an editor's note by Richard Thomason [28, p. 259, FN 9] that $\boxtimes$, "is interpreted in the sense of 'necessarily always'." After all, $\boxtimes\varphi$ is true in a world w at a time x just in case φ is true in all worlds at all times, quantifying over both the modal and temporal dimensions of Montague's semantics at once.

By letting a *moment* be any ordered pair $\langle w, x \rangle \in W \times T$, the semantics for $\boxtimes$ quantifies over all moments.²³ By contrast, I will take the following *universal semantics* for $\boxplus$ to quantify over all worlds while leaving the temporal parameter unchanged:

- $(\boxplus) \mathcal{M}, w, x \models \boxplus\varphi \text{ iff } \mathcal{M}, u, x \models \varphi \text{ for all } u \in W.$

²³A moment here is a pair whose first coordinate is a primitive world. By constructing possible worlds in §3, the world coordinate is a function from durations to world states so that a moment fixes the world state that the possible world occupies at that duration from the origin in that possible world.

To evaluate whether $\Box$ or $\Box$ expresses metaphysical necessity, I will take $\mathcal{L}^K$ to be the result of replacing $\Box$ in $\mathcal{L}^M$ with $\Box$ while maintaining the other operators included in the language. Next we may derive the semantic clause for Δ from its definition above:

$$(\Delta) \mathcal{M}, w, x \models \Delta\varphi \text{ iff } \mathcal{M}, w, y \models \varphi \text{ for all } y \in T.$$

Extending $\mathcal{L}^K$ to include $\Box$ as a primitive symbol makes $\Box\varphi \leftrightarrow \Box\Delta\varphi$ valid by being true in every world at every time on any two-dimensional model, thereby rendering $\Box$ redundant. Rather, I will maintain $\Box\varphi := \Box\Delta\varphi$ as a metalinguistic abbreviation in $\mathcal{L}^K$, deriving the semantics for $\Box$ from its definition. Since **T3** in the *Appendix* proves that $\Box$ cannot be defined in $\mathcal{L}^M$, it follows that $\mathcal{L}^K$ is more expressive than $\mathcal{L}^M$. For the purposes of comparing $\Box$ and $\Box$ in a common language, $\mathcal{L}^K$ enjoys a clear advantage over working in $\mathcal{L}^M$. Assuming that $\mathcal{L}^M$ is to be replaced by $\mathcal{L}^K$ where $\Box$ is then defined in terms of $\Box$ and Δ as above, the question remains whether or not to read ' $\Box\varphi$ ' as 'It is necessarily the case that φ ' in accordance with Thomason's suggested reading of ' $\Box\varphi$ ' as 'It is necessarily always the case that φ ', or to follow Montague.²⁴

In opposition to Thomason's reading, one might appeal to the invalidity of **P1** and **P2** over the class of two-dimensional models in defense of taking $\Box$ and $\Diamond\varphi := \neg\Box\neg\varphi$ to be the metaphysical modals rather than $\Box$ and $\Diamond\varphi := \neg\Box\neg\varphi$.²⁵ So long as there are multiple times in T , we may consider a model $\mathcal{M}_2$ where p_i is assigned to a set of world-time pairs that includes $\langle w', x \rangle$ for all $w' \in W$ at a fixed time $x \in T$ but that does not include $\langle w, x' \rangle$ for all times $x' \in T$ at a fixed world $w \in W$. It follows that $\Box p_i$ may be true in w at x while Δp_i is false, thereby invalidating **P1** where **P2** is equivalent. By contrast, the following *trivial perpetuity principles* which result from replacing $\Box$ and $\Diamond$ in **P1** and **P2** with $\Box$ and $\Diamond$ may be shown to be valid:

$$\mathbf{TP} \Box\varphi \rightarrow \Delta\varphi.$$

$$\mathbf{CT} \nabla\varphi \rightarrow \Diamond\varphi.$$

Given the definition $\Box\varphi := \Box\Delta\varphi$, **TP** is an instance of the **T** axiom $\Box\psi \rightarrow \psi$ where **CT** is equivalent. Instead of expressing substantive interaction principles for tense and modality, the trivial perpetuity principles are valid over the two-dimensional models of $\mathcal{L}^K$ for entirely modal reasons. To avoid appealing to the status of the operators in one language or another, we may put the point purely semantically as follows:

$$\begin{aligned} \text{Triviality: } & (\forall w \in W \forall x \in T : \mathcal{M}, w, x \models \varphi) \rightarrow (u \in W \rightarrow \forall x \in T : \mathcal{M}, u, x \models \varphi). \\ & (u \in W \wedge \exists x \in T : \mathcal{M}, u, x \models \varphi) \rightarrow (\exists w \in W \exists x \in T : \mathcal{M}, w, x \models \varphi). \end{aligned}$$

The principles above are instances of the first-order theorems $\forall w \Psi \rightarrow \Psi[u/w]$ and $\Psi[u/w] \rightarrow \exists w \Psi$.²⁶ Despite including quantification over times, nothing in Ψ accounts for why the *Triviality* principles are valid. Rather, these principles are valid on account of universal instantiation and existential generalization with respect to quantification

²⁴ Admittedly, it is not clear that Montague [1] was concerned with metaphysical modality, though the semantics he provides for $\Box$ is completely unrestricted in quantifying over times and worlds.

²⁵ Dorr and Goodman [29, p. 636] present considerations along these lines, though they, "think it best to avoid 'world'-talk altogether in theorizing about temporary matters" (p. 646). In opposition to this perspective, I will assume that the semantic primitives included in the models of a language ought to provide the intuitive bedrock by which to define meaningful truth-conditions for the sentences of the language.

²⁶ Replace Ψ with either: (1) $w \in W \rightarrow \forall x \in T : \mathcal{M}, w, x \models \varphi$; or (2) $w \in W \wedge \exists x \in T : \mathcal{M}, w, x \models \varphi$.

over the worlds in W independent of the quantification over times in Ψ . That **TP** and **CT** are valid over the two-dimensional models is no compensation for being valid for the wrong reason. By contrast, the universal semantics for $\boxdot$ and $\boxplus$ enjoys a clear advantage over the Montagovian semantics for $\boxtimes$ by making **P1** and **P2** substantive interaction principles by virtue of their greater expressive power.

As Dorr and Goodman [29, pp. 635, 655] observe, the universal necessity operator $\boxdot$ can be defined in terms of the Montagovian operator $\boxtimes$ given the resources that Fine [30] provides. By including a countable set of *time variables* $\mathcal{V} := \{t_i : i \in \mathbb{N}\}$ in the language which may be bound by first-order quantifiers and taking an *assignment* to be any function $g : \mathcal{V} \rightarrow T$ from time variables in $\mathcal{V}$ to times in T , consider:

($\exists t$) $\mathcal{M}, w, x, g \models \exists t\varphi$ iff $\mathcal{M}, w, x, g' \models \varphi$ for some g' differing from g at most in t .

(Pr) $\mathcal{M}, w, x, g \models \text{Present}(t)$ iff $g(t) = x$.²⁷

Converse Definition: $\boxdot\varphi := \exists t[\text{Present}(t) \wedge \boxtimes(\text{Present}(t) \rightarrow \varphi)]$.²⁸

By taking $\mathcal{L}^F$ to extend $\mathcal{L}^M$ to include the Finean resources given above, **L4** in the *Appendix* shows that $\boxdot\varphi$ is true in world w at time x on assignment g just in case φ is true in world u at time x on assignment g for all worlds $u \in W$. Given this alignment with the universal semantic clause provided above for $\boxdot$, **T4** shows that the perpetuity principles are invalid over the two-dimensional models for $\mathcal{L}^F$.

Whereas $\boxtimes$ is easy to define in terms of $\boxdot$ and $\triangle$ in $\mathcal{L}^K$, the ideological complexity of the *Converse Definition* reflects the relative obscurity of $\boxdot$ from the perspective of the primitive expressions included in $\mathcal{L}^F$. Given the naturalness of the semantics for $\boxdot$, the complexity of the *Converse Definition* indicates how awkward it is to make do with the operator $\boxtimes$ whose semantic clause quantifies over moments in $W \times T$ instead of worlds in W . Although the extra ideology adds expressive power to $\mathcal{L}^F$, including temporal predicates and quantifiers over times abandons Prior's insight that much of the trouble that McTaggart [23] faced was due to cobbling together tense operators, temporal predicates, and first-order quantification over times all into one language. Following Prior, I will exclude temporal predicates and first-order quantification over times from the object language, limiting use of these resources to the metalanguage where I will articulate truth-conditions for the sentences of a bimodal language. In addition to avoiding the threat of making the temporal operators superfluous, this approach does not carry the same commitment to an ontology of times.

Instead of validating the perpetuity principles trivially by taking the semantics for the metaphysical modals to quantify over all worlds and times, the following subsection considers attempts to validate the perpetuity principles by constraining the class of two-dimensional models. As I will argue, the resulting theory is committed to an absolute theory of time that cannot be avoided without undermining the significance of the truth-conditions for the sentences of the language. This will set the stage for

²⁷Dorr and Goodman [29, p. 637] admit that Fine's [30] use of propositional quantifiers to eliminate time variables from the expanded language behaves pathologically when $\boxdot$ is replaced by $\boxtimes$.

²⁸Dorr and Goodman [29, p. 656] refer to $\boxdot$ here as *immediate necessity* and are clear that they nowhere endorse model-theoretic characterizations of metaphysical necessity, Montagovian or otherwise.

§3 which provides the task semantics, proving that **P1** and **P2** are valid without imposing any *ad hoc* constraints on the space of models for the language.

2.3 Absolute Time

Instead of taking the semantics for the metaphysical modals to quantify over moments in $W \times T$, Kaplan’s [2] semantics for $\Box$ quantifies over just the worlds W in a two-dimensional model of $\mathcal{L}^K$. Although Kaplan does not articulate a logic for his two-dimensional semantics, **P1** and **P2** are invalid over the two-dimensional models that Kaplan defines. Since **T5** shows that no frame constraint on $\langle W, T, \leq \rangle$ can validate the perpetuity principles without collapsing the temporal order to include at most one time, the only way to validate the perpetuity principles over the two-dimensional semantics is to impose a model constraint. In particular, **T6** shows that **P1** and **P2** are valid over the abundant two-dimensional models of $\mathcal{L}^K$ which I will define as follows:

Time-Shift: The worlds $w, w' \in W$ are *time-shifted from x to y* — i.e., $w \approx_x^y w'$ — in a model $\mathcal{M}_2$ of $\mathcal{L}^K$ iff there is an order automorphism $\bar{a} : T \rightarrow T$ where $y = \bar{a}(x)$ and the following holds for all sentence letters $p_i \in \mathbb{L}$ and times $z \in T$:

$$\langle w', \bar{a}(z) \rangle \in |p_i| \Leftrightarrow \langle w, z \rangle \in |p_i|.^{29}$$

Abundance: A two-dimensional model $\mathcal{M}_2$ of $\mathcal{L}^K$ is *abundant* iff for every $w \in W$ and $x, y \in T$, there is some $w' \in W$ that is time-shifted from x to y , i.e., $w \approx_x^y w'$.

Abundant models include all time-shifted worlds.³⁰ Paradigm examples of abundant models identify T with either the set of integers $\mathbb{Z}$, rational numbers $\mathbb{Q}$, or real numbers $\mathbb{R}$. As **T7** shows, time is unbounded in abundant models with at least two distinct times.³¹ Nevertheless, it is natural to take some systems such as a game of chess to have a bounded set of times. That abundant models cannot accommodate bounded temporal orders is a significant limitation which the task semantics avoids.

Restricting to the abundant models validates the perpetuity principles, but by a means formally weaker than the frame constraints of correspondence theory. A frame constraint restricts the structure $\langle W, T, \leq \rangle$ alone and so holds under every valuation, whereas *Abundance* is defined through the time-shift relation $\approx_x^y$ and so constrains the admissible valuations rather than the frame itself. By **T5** this is unavoidable: no condition on $\langle W, T, \leq \rangle$ validates **P1** and **P2** short of collapsing the temporal order, and so the perpetuity principles correspond to no frame constraint at all. In giving up the correspondence between the perpetuity axioms and a condition on the frames, we give up the central explanatory virtue of relational semantics— that a logic’s theorems are underwritten by, and hold across all valuations of, the structure provided by the

²⁹ An *order automorphism* on the structure $\langle T, \leq \rangle$ is a monotonic bijection $\bar{a} : T \rightarrow T$ where $\bar{a}(x) \leq \bar{a}(y)$ whenever $x \leq y$, shifting all times in T without changing their order.

³⁰ Not all models are abundant: letting $W = \{w\}$ and $T = \{0, 1\}$ where $|p_1| = \{\langle w, 0 \rangle\}$, there is no automorphism on T mapping 0 to 1 that preserves truth-values, and so the model is not abundant.

³¹ A weak partial order $\langle T, \leq \rangle$ is *bounded* if both of the following hold and *unbounded* if neither holds:

Bounded Below: There is some $y \in T$ where $y \leq x$ for all $x \in T$.

Bounded Above: There is some $y \in T$ where $x \leq y$ for all $x \in T$.

frame.³² In addition to this formal deficit, abundant models are required to include a vast set of primitive worlds. We may state this precisely as follows:

Temporary: A sentence φ of $\mathcal{L}^K$ is *temporary* in a two-dimensional model $\mathcal{M}_2$ iff $\mathcal{M}_2, w, x \models \varphi$ and $\mathcal{M}_2, w, y \not\models \varphi$ for some world w and times x and y .

Abundant models with temporary sentences include all merely temporal differences between worlds. For instance, given an abundant model $\mathcal{M}_2$ in which φ is temporary, there is a world w and times $x, y \in T$ where $\mathcal{M}_2, w, x \models \varphi$ and $\mathcal{M}_2, w, y \not\models \varphi$, and so by *Abundance* there is a world w' that is time-shifted $w \approx_x^y w'$ from x to y . As shown in **L6**, what is true in w at x is exactly the same as what is true in w' at y . Since $w \approx_x^y w'$, it follows more generally that there is some order automorphism $\bar{a} : T \rightarrow T$ where $w \approx_z^{\bar{a}(z)} w'$ for all $z \in T$ by **L5**. Again by **L6**, it follows that for every time $z \in T$ exactly the same sentences that are true in w at z are true in w' at $\bar{a}(z)$. Insofar as there are intended models where W represents a meaningful range of possible histories of events without ever representing the same history more than once, it follows that abundant models with temporary sentences include all merely temporal differences between otherwise indistinguishable histories of events.

I will take *temporal absolutism* to be the claim that there are merely temporal differences between otherwise indistinguishable histories— $w \approx_x^y w'$ for some $w \neq w'$ —a thesis that I will assume it is natural to resist. This formal deficit has an ontological cost, since requiring the models of $\mathcal{L}^K$ to be abundant gives rise to an unfortunate trade-off: either one must embrace temporal absolutism or else admit an excess of possible worlds that represent the same possible history many times over. Although it is preferable to avoid redundancies among the semantic primitives included in a model, one might attempt to defend a restriction to the abundant models by taking the time-shifted worlds to be empty artifacts. However, insofar as sets of world-time pairs are to provide meaningful truth-conditions for the well-formed sentences of $\mathcal{L}^K$, the worlds in W cannot be entirely void of significance. For instance, an *abundance theorist* might attempt to maintain intelligible truth-conditions by abstracting from the merely temporal differences by way of the following definitions:

Time-Shifted Worlds: $w \approx w'$ iff $w \approx_x^y w'$ for some $x, y \in T$.

World Abstraction: $[w] := \{w' \in W \mid w \approx w'\}$.

Possible Worlds: $W_\approx := \{[w] \subseteq W \mid w \in W\}$.

By letting $[w]$ be the set of time-shifted worlds that represent the same possible world as w , an abundance theorist might claim that $W_\approx$ represents the range of genuinely distinct possible worlds rather than the primitive set W . Although an abundant model which admits temporary sentences is guaranteed to include merely temporal differences between the primitive worlds in W , the elements in $W_\approx$ abstract from the merely temporal differences between worlds. An abundance theorist may take **T6** to show how

³²The frame-incompleteness of *Abundance* is an instance of the general problem that interaction axioms in products of modal logics pose for frame definability (see Kurucz [31, p. 221]). By contrast, compare the correspondence theorems for the tense fragment given in §3.3 following van Benthem [32].

to validate **P1** and **P2** by restricting consideration to the abundant models without embracing absolutism or undermining the interpretation of possible worlds.

Despite providing a method for identifying a reduced range of genuinely distinct possible worlds for each abundant model of $\mathcal{L}^K$, it is important to observe that $W_\approx$ is defined in terms of the interpretation of the sentence letters of the language. Even if the worlds $w \approx w'$ are time-shifted in a model $\mathcal{M}_2$ of $\mathcal{L}^K$ and so $[w] = [w']$, it does not follow that $w \approx w'$ in any other model $\mathcal{M}'_2$ of $\mathcal{L}^K$. For instance, assuming $w \approx_x^y w'$ in $\mathcal{M}_2$ for just $x, y \in T$ where $x \neq y$ and $\langle w, x \rangle \in |p_0|$, there is an automorphism $\bar{a} : T \rightarrow T$ where $y = \bar{a}(x)$ and $\langle w', y \rangle \in |p_0|$. By letting $\mathcal{M}'_2$ be identical to $\mathcal{M}_2$ except for taking $\langle w', y \rangle \notin |p_0|$, it follows that $w \not\approx_x^y w'$ in $\mathcal{M}'_2$ for any $x, y \in T$, and so $[w] \neq [w']$ in $\mathcal{M}'_2$. This construction demonstrates how the equivalence classes in $W_\approx$ may expand or contract depending on the truth-conditions assigned to the sentence letters by a model $\mathcal{M}_2$ of the language $\mathcal{L}^K$. The same construction shows that *Abundance* itself may be destroyed by changing the interpretation of a single sentence letter, confirming that it is a condition on models rather than frames since it is not preserved under a change of valuation. As a result, an abundance theorist cannot appeal to $W_\approx$ in order to specify meaningful truth-conditions for the sentence letters of $\mathcal{L}^K$ without circularity. In particular, consider the following definitions:

Time-Shifted Moments: $[w, x] := \{\langle u, y \rangle \mid w \approx_x^y u\}$.

Truth-Conditional Abstraction: $|p_i|_\approx := \{|w, x] \mid \langle w, x \rangle \in |p_i|\}$.

The first definition identifies the genuinely distinct moments in $\mathcal{M}_2$ by abstracting from the differences between moments which make the same sentence letters true in $\mathcal{M}_2$. However, for $|p_i|_\approx$ to provide a meaningful truth-condition for p_i in $\mathcal{M}_2$, one must already have an independent grasp of the original truth-condition $|p_i|$ provided by $\mathcal{M}_2$, undermining the need to specify $|p_i|_\approx$ in the first place. In addition to circularity, taking the truth-conditions for the sentence letters of $\mathcal{L}^K$ to be the basis upon which to identify the range of genuinely distinct possible worlds $W_\approx$ puts the cart before the horse. Rather, the *intended models* of the language aim to provide a range of meaningful semantic primitives by which to specify truth-conditions for the language, not the other way around. Whereas an *instrumentalist* gives up the ambition to admit intended models, I will take it to be an advantage of any semantic theory to avoid undermining the interpretation of the semantic primitives for that theory.

It is worth comparing a similar strategy applied to the semantics of a first-order extensional language $\mathcal{L}^1$. Given a domain D that represents a meaningful range of primitive *objects*, a truth-conditional semantics may interpret $\mathcal{L}^1$ by specifying the extensions of the constants and n -place predicates as elements of D and subsets of D^n . For instance, we may interpret a constant c in a model by assigning it to an element of D and interpret a one-place predicate F in a model by taking its extension to be a subset of the domain D . However, if it is denied that D includes genuinely distinct objects but rather some other entities which may represent the same genuinely distinct object many times over, we lose our initial grasp on the meaning of the constant c and the predicate F . If D is left uninterpreted, or else interpreted in terms of another

assignment of constants and predicates to extensions in D^n , then we cannot look to D to provide an independent basis upon which to interpret the language $\mathcal{L}^1$.

In keeping with a standard methodology in truth-conditional semantics, I will take the semantic primitives in an intended model for a given object language to provide an independently meaningful basis upon which to make sense of the truth-conditions for that language. Rather than presuming that the only way for the semantic primitives in an intended model to be meaningful is to be identical to the parts of the reality that they model, I will take the intended models of a language to provide an idealization that simulates what that object language is able to express with the resources of a well-theorized metalanguage. Whereas the object language is of primary metaphysical significance, the metalanguage provides expressive resources for simulating what the object language is able to express rather than a description of the entities to which the object language is implicitly committed. I will refer to this approach as *simulation metasemantics* in opposition to *realist metasemantics* which takes the intended model to include the constituents of reality that the object language describes as well as to *instrumentalist metasemantics* in which the semantic primitives have no significance beyond the instrumental role that they play in the semantics.³³

To validate the perpetuity principles without undermining the significance of the truth-conditions for the language, the following section presents an alternative to two-dimensional semantics that is compatible with a simulation metasemantics. Instead of taking possible worlds to be primitive points devoid of any internal structure, I will provide a *task semantics* that constructs possible worlds from *world states*, *tasks*, and *durations*. As I show, the resulting models provide a natural first-order simulation of what the bimodal language is able to express while also validating **P1** and **P2**.

3 Possible Worlds

To interpret the bimodal language $\mathcal{L} := \langle \mathbb{L}, \perp, \rightarrow, \Box, \Box, \Box \rangle$, well-formed sentences will be evaluated at a *possible world* and a *duration* in addition to a model of $\mathcal{L}$. Rather than following Montague [1] and Kaplan [2] in taking possible worlds to be primitive, I will define possible worlds to be functions from durations to world states.³⁴ Whereas world states are interpreted as the instantaneous maximal possible configurations of the system under study, the durations parameterize the possible trajectories through the space of world states. Accordingly, I will take the durations and not the world states to form a total order while also positing additive group structure so that durations can be added and subtracted as is natural. Given the parameterization provided by a possible world, a duration x may be used to indicate the time after x duration from the origin in that possible world, where this justifies the convention of referring to durations as *times* in a possible world. The models of $\mathcal{L}$ will include an interpretation

³³Cresswell [33] shows that tense operators supplemented with selective indexical devices— operators that store and retrieve evaluation indices— are expressively equivalent to first-order quantification over times, extending the result to modal operators and possible worlds. Cresswell concludes that “the [ontological] commitment is genuine if the range of sentences in which these constructions occur requires a semantics equivalent in power to explicit variable binding” (p. 2). The bimodal language $\mathcal{L}^K$ includes no such selective devices, and its operators are all unselective in Cresswell’s sense. However, even if $\mathcal{L}^K$ were enriched with such devices, I do not follow Cresswell in taking semantics to be any guide to ontological commitment.

³⁴Although the world states will be primitive for our purposes, I define world states in [6] by appealing to the task and parthood relations to provide a semantics for counterfactual conditional operators.

function that maps each sentence letter to a set of world states which provides the truth-condition for that sentence. Since each set of world states represents the ways for the system to be without including any temporal elements, durations are strictly exogenous to the interpretation of the sentence letters in language $\mathcal{L}$.

Positing a set of world states W in addition to a set of durations D separates the *what* from the *when* which Prior [21] and Thomason [25] run together, as brought out in §2.1. The truth-condition for a sentence letter of $\mathcal{L}$ specifies all the configurations of the system in which that sentence is true rather than merely specifying the times at which it happens to be true. To add and subtract durations without positing any additional structure, I will take a *temporal order* to be a totally ordered abelian group $\mathcal{D} = \langle D, +, 0, \leq \rangle$.³⁵ Instead of admitting the incomparable times permitted by a partial order, $\mathcal{D}$ will be used to define all possible totally ordered possible worlds. Since not every path through the space of world states qualifies as a possible world, I will equip frames with a *parameterized task relation* $w \Rightarrow_x u$ to encode which transitions between world states $w, u \in W$ over a duration $x \in D$ are possible for a given system. The task relation abstracts from the universal laws that one might hope to articulate for that system to describe and predict which transitions are possible. For instance, specifying the rules of chess determines the extension of the task relation where any game of chess must begin with the initial board state and all transitions between board states conform to the rules. More generally, a *task frame* is any $\mathcal{F} = \langle W, \mathcal{D}, \Rightarrow \rangle$ where W is a nonempty set of *world states*, $\mathcal{D}$ is a temporal order, and $\Rightarrow$ satisfies the constraints:

Nullity: $w \Rightarrow_0 w$.

Reflection: If $w \Rightarrow_x u$, then $u \Rightarrow_{-x} w$.

Compositionality: If $w \Rightarrow_x u$ and $u \Rightarrow_y v$, then $w \Rightarrow_{x+y} v$.

The constraints above require every world state to transition to itself in zero duration, every task to be invertible by flipping the sign of the duration, and all consecutive tasks to compose.³⁶ Each task frame provides what is also called a *non-deterministic dynamical system*.³⁷ Equivalently, a task frame is a labeled transition system (LTS) where transitions between world states are labeled by durations. Postponing further consideration of these connections to §4.2, I will define a *world history* to be a function $\tau : X \rightarrow W$ where $X \subseteq D$ is a nonempty convex set and $\tau(x) \Rightarrow_{y-x} \tau(y)$ for all times $x, y \in X$.³⁸ Whereas $\mathcal{D}$ is required to be infinite and unbounded, X may be finite and

³⁵A *group* is any $\langle G, \cdot, 1_G \rangle$ where: (1) $a \cdot b \in G$ whenever $a, b \in G$; (2) $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ for all $a, b, c \in G$; (3) $1_G \in G$ where $a \cdot 1_G = 1_G \cdot a = a$ for all $a \in G$; and (4) for each $a \in G$, there is some $-a \in G$ where $a \cdot (-a) = 1_G$, written $a - a = 1_G$ for ease. A group is *abelian* just in case $a \cdot b = b \cdot a$ for all $a, b \in G$. A group is *totally ordered* by $\leq$ just in case $\leq$ is a total order where for all $a, b, c \in G$, if $a \leq b$, then $a \cdot c \leq b \cdot c$.

³⁶Letting $(w)_x := \{u \in W : w \Rightarrow_y u \text{ where } |y| < x\}$, the set $B_{\mathcal{F}} := \{(w)_x : w \in W, x \in D, \text{ and } x > 0\}$ generates a topology $\mathcal{T}_{\mathcal{F}} := \langle W, \mathcal{O}_{\mathcal{F}} \rangle$ where the set of *open sets* $\mathcal{O}_{\mathcal{F}}$ is the result of closing $B_{\mathcal{F}}$ under arbitrary union and finite intersection. Since *Reflection* ensures $u \in (w)_x$ if and only if $w \in (u)_x$ for all $x > 0$, the specialization preorder is symmetric, so $\mathcal{T}_{\mathcal{F}}$ is always R0. See T14 and T13 in the *Appendix*.

³⁷A more common notation with a sparser theory of *positive durations* in place of a temporal order takes a non-deterministic dynamical system to be any $\mathbb{D} = \langle X, \mathcal{D}, \{R_t\}_{t \in D} \rangle$ where $\mathcal{D} = \langle D, +, 0 \rangle$ is a monoid and $R_t \subseteq X \times X$ satisfies $R_0 = 1_X$ and $R_{s+t} = R_s \circ R_t$ for all $s, t \in D$.

³⁸A subset $X \subseteq D$ is *convex* just in case $y \in X$ whenever $x, z \in X$ and $x < y < z$.

bounded.³⁹ For instance, whereas we may draw on $\mathbb{Z}$ to provide durations, individual games of chess have initial states as well as final states for finite games.

Although the durations in D form a total order so that no time in any world history occurs more than once, world histories may assign multiple durations to the same world state. By permitting world states to occur at more than one time in a world history, there may not be a single answer to the question which world states came before or after another world state in a given world history. Rather, it is by specifying a duration $x \in D$ that we may determine which world states are in the past or future relative to the time after that duration from the origin in the world history in question. In the case of a meandering end game in chess, there is nothing to prevent the chessboard from occupying the same board state at more than one time. A *moment*, or equally a *time*, is a pair $\langle \tau, x \rangle$ consisting of a world history τ and a duration $x \in D$, where $\tau(x)$ returns the world state that the history occupies at the duration x from the origin within τ . Whereas the durations are primitive, a moment is derived, depending on both a world history and a duration. The pair $\langle \tau, x \rangle$ fixes *when* the point of evaluation falls, and the world state $\tau(x)$ fixes *what* obtains at that time, so nothing prevents the same world state from occurring at many times in a single world history, in contrast with the strictly ordered world states of §2.1. Referring to the duration x alone as a time is therefore shorthand which leaves the world history of evaluation implicit, as in the convention adopted above.

Instead of positing an abundance of primitive time-shifted possible worlds, the same semantic primitives included in $\mathcal{F}$ generate an abundance of world histories. Letting $H_{\mathcal{F}}$ be the set of all world histories defined over the frame $\mathcal{F}$, we may take the semantic clauses for the modal operators to quantify over all world histories in $H_{\mathcal{F}}$. Certain applications may restrict consideration to the *complete world histories* $H_{\mathcal{F}}^* := \{\tau \in H_{\mathcal{F}} \mid \text{dom}(\tau) = D\}$ which assign all times in D to world states in W or, alternatively, to the *length n world histories* $H_{\mathcal{F}}^n := \{\tau \in H_{\mathcal{F}} : |\text{dom}(\tau)| = n\}$. Although there are many restricted sets of world histories that one might consider for a given dynamical system, the metaphysical modals concern the broadest set $H_{\mathcal{F}}$ provided by the frame for the system in question. Letting $\tau \approx_x^y \sigma$ indicate there is an order automorphism $\bar{a} : D \rightarrow D$ which time-shifts τ to σ so that $\text{dom}(\sigma) = \bar{a}(\text{dom}(\tau))$, $y = \bar{a}(x)$, and $\tau(z) = \sigma(\bar{a}(z))$ for all $z \in \text{dom}(\tau)$, we may let $\tau \approx \sigma$ express that $\tau \approx_x^y \sigma$ for some $x, y \in D$. Since $\mathcal{D}$ is translation invariant, each time $x \in D$ induces an automorphism $\bar{a}(z) = z + x$ which shifts the temporal order forwards by x . We may then take the set $[\tau]_{\mathcal{F}} := \{\sigma \in H_{\mathcal{F}} \mid \tau \approx \sigma\}$ of world histories time-shifted from τ to represent a *possible world* where $\mathbb{W}_{\mathcal{F}} := \{[\tau]_{\mathcal{F}} \mid \tau \in H_{\mathcal{F}}\}$ is the set of all possible worlds defined over the frame $\mathcal{F}$ for a given system. Whereas possible worlds correspond to distinct paths through the space of all world states W , world histories parameterize those paths by fixing an assignment of times to world states while acknowledging that the choice of origin does not represent anything significant. Although the set of possible worlds may claim to hold a metaphysical standing that the set of world histories cannot, it is the world histories that will play an important

³⁹Every nontrivial ordered group is infinite and unbounded: given $d > 0$, translation invariance yields the strictly ascending chain $0 < d < 2d < 3d < \dots$, and inverses give $\dots < -3d < -2d < -d < 0$.

role in the semantics. Since possible worlds will play no further role throughout what follows, I will refer to world histories as possible worlds for familiarity.

Having presented the construction of possible worlds, we may define the *models* of $\mathcal{L}$ to be any tuple $\mathcal{M} = \langle W, \mathcal{D}, \Rightarrow, |\cdot| \rangle$ where $\mathcal{F} = \langle W, \mathcal{D}, \Rightarrow \rangle$ is a task frame and $|p_i| \subseteq W$ for every sentence letter $p_i \in \mathbb{L}$. The well-formed sentences of $\mathcal{L}$ are evaluated at a possible world $\tau \in H_{\mathcal{F}}$ and time $x \in D$ in a model $\mathcal{M}$ as follows:

- (p_i) $\mathcal{M}, \tau, x \models p_i$ iff $x \in \text{dom}(\tau)$ and $\tau(x) \in |p_i|$.
- ($\perp$) $\mathcal{M}, \tau, x \not\models \perp$.
- ($\rightarrow$) $\mathcal{M}, \tau, x \models \varphi \rightarrow \psi$ iff $\mathcal{M}, \tau, x \not\models \varphi$ or $\mathcal{M}, \tau, x \models \psi$.
- ($\Box$) $\mathcal{M}, \tau, x \models \Box\varphi$ iff $\mathcal{M}, \sigma, x \models \varphi$ for all $\sigma \in H_{\mathcal{F}}$.
- ($\Box$) $\mathcal{M}, \tau, x \models \Box\varphi$ iff $\mathcal{M}, \tau, y \models \varphi$ for all $y \in D$ where $y < x$.
- ($\Box$) $\mathcal{M}, \tau, x \models \Box\varphi$ iff $\mathcal{M}, \tau, y \models \varphi$ for all $y \in D$ where $x < y$.

Despite taking possible worlds to be defined rather than primitive points, possible worlds and times play the same conceptual roles that they have traditionally played in bimodal frameworks. Given a possible world together with a time in a model of $\mathcal{L}$, we may determine the truth-value of any sentence in the language. Whereas the semantics for the metaphysical modals quantify over all possible worlds in $H_{\mathcal{F}}$, the semantics for the tense operators quantify over all times in D .⁴⁰ Certain applications may restrict the tense operators to the domain $\text{dom}(\tau)$ for the possible world τ of evaluation or introduce semantic clauses for modal operators that quantify over the complete worlds in $H_{\mathcal{F}}^*$ to avoid discrepancies between the times in each world. For other systems, temporal discrepancies between possible worlds are perfectly appropriate.

In order to get a better sense of the semantics, consider a particular chess game α in which Black nearly checkmates White on move 31 before blundering the dark squared bishop. Playing on until move 47 in α , Black manages to win the end game, saying:

(K) If I hadn't blundered my bishop, I would have won much sooner.

Although the present framework is not equipped to interpret tensed counterfactual conditionals, it is clear that Black is lamenting the existence of another possible game in which the blunder had been avoided.⁴¹ Nevertheless, we may evaluate the following:

(P) Black could have checkmated White much sooner.

Given a sufficiently strong reading of 'could', we may regiment P in $\mathcal{L}$ as $\Diamond\Diamond p_w$ where p_w reads 'White is in checkmate'. The sentence $\Diamond\Diamond p_w$ is true in α at move 47 just in case there is a game β where $\Diamond p_w$ is true at move 47. Although move 47 could not have been played in a game in which $\Diamond p_w$ is true at move 47, the present framework nevertheless permits sentences to be evaluated at move 47 in β . For instance, if p_w is

⁴⁰ Although it is easy to define the *reflexive tense operators* $\Box'\varphi := \Box\varphi \wedge \varphi$ and $\Box'\varphi := \Box\varphi \wedge \varphi$ in terms of the *irreflexive tense operators* $\Box$ and $\Box$ given above, the irreflexive tense operators cannot be defined in terms of the reflexive tense operators. The proof requires a bisimulation argument, and will be omitted here.

⁴¹ I develop a hyperintensional semantics for tensed counterfactual conditionals in [6].

true at move 31 in β , then $\Diamond p_w$ is true in β at move 47. By contrast, every sentence letter is false in β at move 47 given that β is not defined at move 47.⁴²

Having provided a theory of truth for the language $\mathcal{L}$, it remains to provide a theory of logical consequence by which to survey the valid forms of reasoning warranted by the semantics. I will take the definition to assume the following standard form:

Logical Consequence: $\Gamma \models \varphi$ iff for any model $\mathcal{M}$ of $\mathcal{L}$, possible world $\tau \in H_{\mathcal{F}}$, and time $x \in D$, if $\mathcal{M}, \tau, x \models \gamma$ for all $\gamma \in \Gamma$, then $\mathcal{M}, \tau, x \models \varphi$.

A well-formed sentence φ of $\mathcal{L}$ is *valid* just in case $\emptyset \models \varphi$, dropping set notation for convenience. In addition to providing an intuitive and general framework for semantic theorizing, the present account validates a simple and strong logic for $\mathcal{L}$ without imposing any frame constraints. As **C2** shows, the perpetuity principles are valid:

P1 $\Box\varphi \rightarrow \Delta\varphi$.

P2 $\nabla\varphi \rightarrow \Diamond\varphi$.

Suppose for contradiction that **P1** has a counterinstance, and so for some well-formed sentence φ , it is metaphysically necessary that φ and yet it is sometimes not the case that φ . Put semantically, $\mathcal{M}, \tau, x \models \Box\varphi$ and $\mathcal{M}, \tau, x \not\models \Delta\varphi$, and so $\mathcal{M}, \tau, y \not\models \varphi$ for some $y \in D$. We may then define the possible world $\sigma(z) = \tau(z - x + y)$ so that $\mathcal{M}, \sigma, x \not\models \varphi$. Thus it follows that $\mathcal{M}, \tau, x \not\models \Box\varphi$, contradicting the above. This proves that **P1** is valid where it follows by classical reasoning that **P2** is equivalent.

Instead of admitting countermodels to **P1** and **P2** as in Kaplan's [2] semantics, the present approach validates the perpetuity principles without imposing *ad hoc* model constraints that undermine the significance of the truth-conditions for the language. Not only do all paradigm examples conform to these principles, **P1** and **P2** support an account of metaphysical modality as the strongest objective modality whereby we may insist that φ is not metaphysically necessary if φ ever fails to be the case. Moreover, by defining possible worlds in terms of the world states, tasks, and durations, the present approach maintains the standard semantic clauses for both tense and modal operators without accepting instrumentalism or temporal absolutism about a primitive ontology of time-shifted worlds. In addition to these merits, taking the metaphysical modals to quantify over possible worlds rather than world-time pairs provides a more expressive theory than Montague's [1] semantics. I will provide further abductive support for the task semantics given above by presenting a logic for $\mathcal{L}$ in §3.2. The following subsection will begin by extending $\mathcal{L}$ to include a stability operator in order to demonstrate the power of the present approach in contrast to two-dimensional semantic theories.

3.1 Restricted Modalities

The semantics for metaphysical necessity $\Box$ quantifies over all possible worlds in $H_{\mathcal{F}}$ defined over the frame $\mathcal{F}$, thereby validating an S5 modal logic. Although this is in keeping with the interpretation of metaphysical modality as the strongest objective modality, there are applications which call for restricted modals. For instance, given a

⁴²Although one could restrict quantification to $\text{dom}(\tau)$ and permute the operators to avoid quantifying outside $\text{dom}(\tau)$, **P21** and **P22** prove that these operators commute in this strongest version of the semantics.

world $\tau \in H_{\mathcal{F}}$ and time $x \in D$, we may let $\langle \tau \rangle_x := \{\sigma \in H_{\mathcal{F}} \mid \sigma(x) = \tau(x)\}$ be the set of possible worlds that intersect τ at x in order to provide the following semantics:

$$(\Box) \quad \mathcal{M}, \tau, x \models \Box\varphi \text{ iff } \mathcal{M}, \sigma, x \models \varphi \text{ for all } \sigma \in \langle \tau \rangle_x.$$

A sentence $\Box\varphi$ is true in a world τ at a time x just in case φ is true at time x in every possible world that occupies the same world state as τ at x .⁴³ Letting $\Diamond\varphi := \neg\Box\neg\varphi$, the *stability operators* $\Box$ and $\Diamond$ may be used to define the following modals:

$$\text{Will Always: } \Box\varphi := \Box\Box\varphi.$$

$$\text{Could Always: } \Box\varphi := \Diamond\Box\varphi.$$

$$\text{Will Eventually: } \Diamond\varphi := \Box\Diamond\varphi.$$

$$\text{Could Eventually: } \Diamond\varphi := \Diamond\Diamond\varphi.$$

Given move number x in a chess game α , the operators above may be used to quantify over the games of chess which occupy the same board state as the game α at time x . Letting p_w be the sentence ‘The white king is in checkmate’, a game of chess may be resigned if the white king will eventually be in checkmate: $\Diamond p_w$. By contrast, letting p_b be the sentence ‘The black king is in checkmate’, a game of chess is still worth playing if each player could eventually win: $\Diamond p_b \wedge \Diamond p_w$.

Another natural restriction on the set of possible worlds arises from considering only those possible worlds which overlap with a given world up to the present time while possibly diverging in the future, and similarly for the past. More precisely:

$$\text{Open Futures: } |\tau\rangle_x := \{\sigma \in H_{\mathcal{F}} \mid \sigma(y) = \tau(y) \text{ for all } y \leq x\}.$$

$$\text{Open Pasts: } \langle \tau|_x := \{\sigma \in H_{\mathcal{F}} \mid \sigma(y) = \tau(y) \text{ for all } y \geq x\}.$$

Whereas $|\tau\rangle_x$ is the set of all possible worlds that occupy the same world state as τ at each time up to and including x while possibly diverging at later times, $\langle \tau|_x$ is the set of all possible worlds that occupy the same world states as τ at x and all later times while possibly diverging at earlier times. Given these definitions, we may introduce operators which quantify over these restricted sets of possible worlds:

$$(\Box) \quad \mathcal{M}, \tau, x \models \Box\varphi \text{ iff } \mathcal{M}, \sigma, x \models \varphi \text{ for all } \sigma \in |\tau\rangle_x.$$

$$(\Box) \quad \mathcal{M}, \tau, x \models \Box\varphi \text{ iff } \mathcal{M}, \sigma, x \models \varphi \text{ for all } \sigma \in \langle \tau|_x.$$

Whereas the *open future* operator $\Box$ quantifies over all possible worlds that agree with the world of evaluation up to the time of evaluation, the *open past* operator $\Box$ is also intelligible and has been included for comparison. Although there is much greater occasion to contemplate the range of open futures at a moment while making plans, there is nothing to stop us from also quantifying over the range of open pasts.

Given that the sets of possible worlds $\langle \tau \rangle_x$, $|\tau\rangle_x$, and $\langle \tau|_x$ are definable in terms of the construction of possible worlds, there is no need to posit additional primitive accessibility relations between possible worlds in order to provide a semantics for the intersection, open future, and open past operators. By contrast, taking possible worlds

⁴³Since $\langle \tau \rangle_x$ is an equivalence class under the relation $\sigma \sim_x \tau := \sigma(x) = \tau(x)$, the monomodal logic of $\Box$ is also S5. However, for non-temporal φ , the truth-value of φ at $\langle \mathcal{M}, \tau, x \rangle$ depends only on $\tau(x)$, so $\varphi \rightarrow \Box\varphi$ is valid, collapsing $\Box$ to the trivial modality on this fragment.

to be structureless points requires each frame to include primitive accessibility relations $R_\times$, $R_\triangleright$, and $R_\triangleleft$ in order to identify the appropriate subsets of possible worlds for these operators to quantify over. Since not all accessibility relations provide appropriate restrictions on the space of possible worlds, a number of further frame constraints would have to be imposed if $\Box$, $\Box$, and $\Box$ are to maintain their intended readings. At the very least, we ought to expect $R_\triangleright$ and $R_\triangleleft$ to be restrictions of $R_\times$ where the intersection of $R_\triangleright$ and $R_\triangleleft$ is nonempty. Even so, merely imposing a range of constraints on the accessibility relations in accordance with the intended readings of the operators does not uniquely determine their extensions. Although permissible, these theoretical costs are avoided entirely by the present theory. Instead of positing a range of frame constraints, the construction of possible worlds makes it provable that $|\tau\rangle_x \subseteq \langle\tau\rangle_x$ and $\langle\tau|_x \subseteq \langle\tau\rangle_x$ where $|\tau\rangle_x \cap \langle\tau|_x = \{\tau\}$. Rather than positing primitive accessibility relations together with a range of constraints, for each fixed duration $x \in D$ we may define $R_\times(\tau, \sigma) := \sigma \in \langle\tau\rangle_x$, $R_\triangleright(\tau, \sigma) := \sigma \in |\tau\rangle_x$, and $R_\triangleleft(\tau, \sigma) := \sigma \in \langle\tau|_x$, avoiding the need to impose frame constraints which align with an intended interpretation. Since $R_\times$, $R_\triangleright$, and $R_\triangleleft$ are definable, we may omit the extra notation entirely.

Rather than accepting the costs of imposing constraints on a primitive accessibility relation between possible worlds, we may extend the present framework to take the *task relation* to be four-place so that $u \Rightarrow_x^w v$ indicates that it is possible for the world state u to transition to the world state v in duration x relative to the world state w . For instance, insofar as the laws of nature are contingent, and perhaps temporary, we may take $u \Rightarrow_x^w v$ just in case it is possible for the world state u to transition to v in duration x while obeying the laws of nature in w . Letting $H_{\mathcal{F}}^w$ be the set of world histories defined over $\Rightarrow^w$, consider the semantics for nomological necessity:

$$(\Box) \quad \mathcal{M}, \tau, x \models \Box\varphi \text{ iff } \mathcal{M}, \sigma, x \models \varphi \text{ for all } \sigma \in H_{\mathcal{F}}^{\tau(x)}.$$

Without assuming $\Rightarrow^w$ to be invariant with respect to the world states $w \in W$, nothing requires $\Box$ to have an S5 logic. By contrast, I will take metaphysical necessity $\Box$ to be the strongest objective modality where an S5 logic is appropriate.⁴⁴ Since the present aim is to develop a bimodal logic for tense and metaphysical modality, I will omit further consideration of the restricted modals $\Box$, $\Box$, $\Box$, and $\Box$.

3.2 Bimodal Logic

Recall the propositional language $\mathcal{L} = \langle \mathbb{L}, \perp, \rightarrow, \Box, \Box, \Box \rangle$ where $\mathbb{L} := \{p_i \mid i \in \mathbb{N}\}$ is a set of *sentence letters* and the *well-formed sentences* of $\mathcal{L}$ are defined as follows:

$$\varphi, \psi ::= p_i \mid \perp \mid \varphi \rightarrow \psi \mid \Box\varphi \mid \Box\varphi \mid \Box\varphi.$$

Letting $\varphi_{\langle \Box \mid \Box \rangle}$ be the result of exchanging all occurrences of $\Box$ and $\Box$ in φ , the *Logic of Tense and Modality* **TM** is the smallest extension of *Classical Propositional Logic* **CPL** to be closed under all instances of the following axiom and rule schemata:

⁴⁴By letting $u \Rightarrow_x v := \exists w(u \Rightarrow_x^w v)$, it follows that $H_{\mathcal{F}}^w \subseteq H_{\mathcal{F}}$ for all $w \in W$, and so $\Box\varphi \rightarrow \Box\varphi$ is valid.

MP	$\varphi, \varphi \rightarrow \psi \vdash \psi.$	TD	<i>If $\vdash \varphi$, then $\vdash \varphi_{\langle P F \rangle}.$</i>
MN	<i>If $\vdash \varphi$, then $\vdash \Box \varphi.$</i>	TK	$\Box(\varphi \rightarrow \psi) \rightarrow (\Box \varphi \rightarrow \Box \psi).$
MK	$\Box(\varphi \rightarrow \psi) \rightarrow (\Box \varphi \rightarrow \Box \psi).$	T4	$\Box \varphi \rightarrow \Box \Box \varphi.$
MT	$\Box \varphi \rightarrow \varphi.$	TB	$\Diamond \top.$
M5	$\Diamond \Box \varphi \rightarrow \Box \varphi.$	TA	$\varphi \rightarrow \Box \Diamond \varphi.$
MF	$\Box \varphi \rightarrow \Box \Box \varphi.$	TL	$(\Diamond \varphi \wedge \Diamond \psi) \rightarrow [\Diamond(\Diamond \varphi \wedge \psi) \vee \Diamond(\varphi \wedge \psi) \vee \Diamond(\varphi \wedge \Diamond \psi)].$

Soundness and completeness are sketched in §5.4 and C3, where these results are implemented in Lean 4 in the associated repository for this paper.⁴⁵ Whereas MP, MN, MK, MT, and M5 are familiar from modal logic, TD makes the logic symmetric with respect to the past and future at each time and TK distributes the future operator over material implication. Additionally, T4 requires the temporal ordering to be transitive and TB asserts that every time has a strictly later time, guaranteeing seriality where the past dual follows by TD. Whereas TA asserts that the present is past to all future times, TL asserts that any two future times are linearly ordered. Given TD, we may also conclude that the present is future to all past times and any two past times are linearly ordered from which it follows that time is linear. Were one to drop the temporal symmetry—for instance, by assuming that there is a first time but no last time—TD must be given up, adding the appropriate duals.

Whereas the axioms and rules discussed so far include either modal or temporal operators, MF is the sole bimodal interaction axiom asserting that what is necessary is necessarily always going to be the case. Since $\Box \Box \varphi \rightarrow \Box \varphi$ is an instance of MT, $\Box \varphi \rightarrow \Box \Box \varphi$ follows from MF where $\Box \varphi \rightarrow \Box \varphi$ follows by TD. Together with MT, these results entail $\Box \varphi \rightarrow (\Box \varphi \wedge \varphi \wedge \Box \varphi)$ which is equivalent to the perpetuity principles:

$$\mathbf{P1} \quad \Box \varphi \rightarrow \Delta \varphi. \qquad \mathbf{P2} \quad \nabla \varphi \rightarrow \Diamond \varphi.$$

This derivation shows that the perpetuity principles follow from MF and MT by classical reasoning. In addition to being stated in primitive terms, MF and MT are easy to justify. We may also derive the following principle:

$$\mathbf{TF} \quad \Box \varphi \rightarrow \Box \Box \varphi.$$

Whereas $\Box \varphi \rightarrow \Box \Box \varphi$ follows by standard modal reasoning, $\Box \Box \varphi \rightarrow \Box \Box \Box \varphi$ and $\Box \Box \Box \varphi \rightarrow \Box \Box \varphi$ are instances of MF and MT. Composing yields $\Box \varphi \rightarrow \Box \Box \varphi$ where $\Box \varphi \rightarrow \Box \Box \varphi$ follows by TD. We may then derive $\Box \varphi \rightarrow (\Box \Box \varphi \wedge \Box \varphi \wedge \Box \Box \varphi)$ where $\Box \varphi \rightarrow \Box(\Box \varphi \wedge \varphi \wedge \Box \varphi)$ follows by modal reasoning. Given the definitions, we have:

$$\mathbf{P3} \quad \Box \varphi \rightarrow \Box \Delta \varphi. \qquad \mathbf{P4} \quad \Diamond \nabla \varphi \rightarrow \Diamond \varphi.$$

⁴⁵See <https://github.com/benbrastmckie/BimodalLogic> for the Lean 4 implementation.

Since $\Box\Diamond\varphi \rightarrow \Diamond\varphi$ is an instance of **MT**, it follows from **TK** that $\Box\Box\Diamond\varphi \rightarrow \Box\Diamond\varphi$. We may then derive $\Diamond\varphi \rightarrow \Box\Diamond\varphi$ from **M5**, where $\Box\Diamond\varphi \rightarrow \Box\Box\Diamond\varphi$ is an instance of **TF**. Thus $\Diamond\varphi \rightarrow \Box\Diamond\varphi$ where $\Diamond\varphi \rightarrow \Box\Diamond\varphi$ follows by **TD**, and so $\Diamond\varphi \rightarrow (\Box\Diamond\varphi \wedge \Diamond\varphi \wedge \Box\Diamond\varphi)$ which is equivalent to $\Diamond\varphi \rightarrow \Delta\Diamond\varphi$. Given **P4**, we may derive the following:

$$\mathbf{P5} \quad \Diamond\Box\varphi \rightarrow \Delta\Diamond\varphi.$$

$$\mathbf{P6} \quad \Box\Box\varphi \rightarrow \Box\Delta\varphi.$$

The perpetuity principles **P1** – **P6** begin to characterize the interactions between tense and modality in **TM**. A number of additional theorems will be derived in §5 where the remainder of the present section will strengthen **TM** by imposing additional frame constraints in addition to extending the expressive power of the language.

3.3 Extensions

Since the logic for metaphysical modality already describes the strongest objective modality, it remains to strengthen the tense logic by further constraining the temporal order $\mathcal{D}$. In addition to taking $\mathcal{D}$ to be a totally ordered abelian group, consider:

DISCRETE: For any time $x \in D$, if there is a later time $y > x$, then there is a least later time $y' > x$ where $z \geq y'$ for all $z > x$.⁴⁶

DENSE: For any times $x, y \in D$ where $x < y$, there is a time $z \in D$ where $x < z < y$.

COMPLETE: Every nonempty set $X \subseteq D$ bounded above has a least upper bound.

The frame constraints impose restrictions on the temporal order $\mathcal{D} = \langle D, +, 0, \leq \rangle$ which characterize the following axioms, as proven in **T8**, **T9**, and **T10** respectively.⁴⁷

$$\mathbf{DF} \quad (\Box\varphi \wedge \varphi \wedge \Diamond\top) \rightarrow \Diamond\Box\varphi.$$

$$\mathbf{DN} \quad \Box\Box\varphi \rightarrow \Box\varphi.⁴⁸$$

$$\mathbf{CO} \quad \Delta(\Box\varphi \rightarrow \Diamond\Box\varphi) \rightarrow (\Box\varphi \rightarrow \Box\varphi).$$

Since no temporal order is both discrete and dense, **TM** cannot be extended to include both **DF** and **DN** while maintaining consistency. Letting **TM_f** extend **TM** to include all instances of **DF**, I will take **TM_d** to include all instances of **DN**, and **TM_c** to include all instances of **CO**, where **TM_{dc}** is the minimal extension of **TM_d** and **TM_c**, corresponding to the *continuous* temporal orders that are both dense and complete.⁴⁹ Although one could maintain certain axioms without their temporal duals by dropping **TD**, I will assume the past and future have the same logic.

⁴⁶It follows that if there is an earlier time $y < x$, then there is a greatest earlier time. Since $\mathcal{D}$ is an ordered group, the map $x \mapsto -x$ reverses the order. Given $y < x$, we have $-x < -y$. The **DISCRETE** condition provides an earliest time $z > -x$, and so $-z < x$ is the latest time before x .

⁴⁷Compare van Benthem's [32] correspondence theorems for Kripke semantics.

⁴⁸Density cannot be expressed in terms of the reflexive tense operators $\Box'$ and $\Diamond'$ from footnote 40.

⁴⁹A totally ordered group is *Archimedean* if for any $x, y \in D$ where $x > 0$, there is a positive integer n where $nx > y$. By Hölder's theorem, every Archimedean totally ordered group is abelian. Since every Archimedean discrete temporal order is complete, **CO** is valid over all Archimedean discrete frames. Insofar as all temporal orders of interest are Archimedean, completeness may be assumed to follow from discreteness.

Since there is little sense in arguing for one logic over another independent of an application, the choice between the **TM** extensions turns on the dynamical systems to which the logic is applied. Given a dynamical system, I will take *metaphysical modality* to be the strongest objective modality whose semantics quantifies over all possible worlds in $H_{\mathcal{F}}$ defined over the task frame $\mathcal{F}$ for that system.⁵⁰ Whereas the task relation constrains which functions from durations to world states are taken to be possible worlds, the metaphysical modals are unrestricted in quantifying over the resulting space of possible worlds, in contrast to the strictly weaker operators considered in §3.1. Since every possible world is defined over the temporal order $\mathcal{D}$ provided by the frame, whatever structure the durations happen to have—discrete, dense, complete, or none of these—holds of metaphysical necessity for the system in question. For instance, if $\mathcal{D}$ is dense, then **DN** and its necessitation $\Box(\Box\Box\varphi \rightarrow \Box\varphi)$ are valid over this frame.⁵¹ Since possible worlds are only ever defined over a single frame, there is no modality whose semantics quantifies over possible worlds defined on distinct frames. However, over the unrestricted class of task frames, none of **DF**, **DN**, and **CO** is valid, nor are their negations: it is not *analytic*—true in all models—that time is discrete, dense, or complete, nor that it fails to be. By contrast, **DN** and its necessitation are valid over the class of dense frames, and so it is analytic that time is dense relative to applications restricted to the dense frames. Disputes about the structure of time concern which class of frames, and so which logic is appropriate to the dynamical systems under study, not what is metaphysically contingent *simpliciter*. It is an advantage of the present framework that such disputes can be articulated in $\mathcal{L}$ without further expressive resources, where **TM_F**, **TM_D**, and **TM_C** completely axiomatize the discrete, dense, and complete theories of time **C3**.

Besides the various extensions of **TM** that one may consider, there are a number of further operators by which to extend the expressive power of $\mathcal{L}$. In particular, the *next* $\oplus$ and *previous* $\ominus$ operators have natural applications given a restriction to the discrete frames over which **DF** and its past dual are valid. Consider the semantics:

- ($\oplus$) $\mathcal{M}, \tau, x \models \oplus\varphi$ iff $\mathcal{M}, \tau, y \models \varphi$ for some $y > x$ where $y \leq z$ for all times $z > x$.
- ($\ominus$) $\mathcal{M}, \tau, x \models \ominus\varphi$ iff $\mathcal{M}, \tau, y \models \varphi$ for some $y < x$ where $y \geq z$ for all times $z < x$.

Citing Dana Scott, Prior [21, p. 66] introduces operators for *tomorrow* and *yesterday* which are represented here as $\oplus$ and $\ominus$ though he does not provide a semantics. Instead, Prior focuses on the *metric tense operators* $Pn\alpha$ and $Fn\alpha$ which indicate that α occurs at a distance n from the time of evaluation in either the past or future. To incorporate metric tense operators, $\mathcal{L}$ would need to include singular terms for

⁵⁰Alternatively, one might limit attention to the universal system—the *cosmos*—claiming that metaphysical modality concerns that system alone, though this is not the approach that I will take here.

⁵¹Dorr and Goodman [29, p. 656] express sympathy for an account of metaphysical modality that is able to express theses about the contingency of the structure of time. Within the present framework, one might give voice to such contingency by lifting the convexity constraint on the domains of possible worlds, admitting *irregular worlds* whose domains are merely closed under differences, and introducing a broader operator that quantifies over the irregular worlds, with respect to which **DF**, **DN**, and **CO** may each be contingent. However, defining logical consequence over the irregular worlds makes **DN** and **CO** invalid over the dense and complete frames respectively, collapsing distinctions between the extensions of **TM** drawn above. Thus the contingency of temporal structure is purchased at the cost of the strength of the tense logic.

durations which I will not explore here. By contrast, semantic clauses for the *since* $\triangleleft$ and *until* $\triangleright$ operators that Kamp [34] introduced can be provided as follows:

- ($\triangleleft$) $\mathcal{M}, \tau, x \models \varphi \triangleleft \psi$ iff $\mathcal{M}, \tau, z \models \psi$ for some time $z < x$ where $\mathcal{M}, \tau, y \models \varphi$ for all intermediate times $y \in D$ where $z < y < x$.
- ($\triangleright$) $\mathcal{M}, \tau, x \models \varphi \triangleright \psi$ iff $\mathcal{M}, \tau, z \models \psi$ for some time $z > x$ where $\mathcal{M}, \tau, y \models \varphi$ for all intermediate times $y \in D$ where $x < y < z$.

We may define $\Diamond\varphi := \top \triangleleft \varphi$ and $\Diamond\varphi := \top \triangleright \varphi$, where $\Box\varphi := \neg\Diamond\neg\varphi$ and $\Box\varphi := \neg\Diamond\neg\varphi$, and so adding $\triangleleft$ and $\triangleright$ to the language obviates the need to take $\Box$ and $\Diamond$ to be primitive.⁵² Since there is no intermediate time at which $\perp$ holds, we may also define $\odot\varphi := \perp \triangleright \varphi$ and $\odot\varphi := \perp \triangleleft \varphi$ which require the argument to be true at the successor or predecessor, respectively. In case there is no immediate successor or predecessor, $\odot$ and $\odot$ are equivalent to $\perp$. Whereas the operators $\triangleleft$ and $\triangleright$ have wide ranging applications that presume nothing of the structure of time, $\odot$ and $\odot$ are only worth including in a language used to study discrete systems. Nevertheless, there are many discrete systems for which $\odot$ and $\odot$ are natural to consider. For instance, one might consider the moves in a chess game, clock cycles on a chip, sequences of program executions, or the actions of one or more agents in a multi-agent system.

Introducing the operators $\triangleleft$ and $\triangleright$ whose semantics requires the guard φ to hold for all times since or until the event ψ affords greater expressive power to the language than $\Box$ and $\Diamond$ provide. The discrete, dense, and complete extensions of a base proof system $\mathbf{TM}^+$ for $\mathcal{L}^+ := \langle \mathbb{L}, \perp, \rightarrow, \Box, \triangleleft, \triangleright \rangle$ are provided in §5.5, where T23 shows $\mathbf{TM}^+$ to be a conservative extension of $\mathbf{TM}$ and C3 establishes completeness for $\mathbf{TM}$ and its extensions. The soundness and completeness of $\mathbf{TM}^+$ and its extensions are verified in the Lean 4 repository for this paper. Even so, these systems lack the means by which to cross reference either times or worlds. To lend greater discriminating power to the language, we may include a vector $\vec{v} = \langle v_1, v_2, \dots \rangle$ of stored times and vector $\vec{\mu} = \langle \mu_1, \mu_2, \dots \rangle$ of stored worlds in the point of evaluation to provide the semantic clauses for the *store operators* $\uparrow_{\mathbf{T}}^i$ and $\uparrow_{\mathbf{M}}^i$ and *recall operator* $\downarrow_{\mathbf{T}}^i$ and $\downarrow_{\mathbf{M}}^i$ for each $i \in \mathbb{N}$:

- ($\uparrow_{\mathbf{T}}$) $\mathcal{M}, \tau, x, \vec{v}, \vec{\mu} \models \uparrow_{\mathbf{T}}^i \varphi$ iff $\mathcal{M}, \tau, x, \vec{v}_{[x/v_i]}, \vec{\mu} \models \varphi$.
- ($\downarrow_{\mathbf{T}}$) $\mathcal{M}, \tau, x, \vec{v}, \vec{\mu} \models \downarrow_{\mathbf{T}}^i \varphi$ iff $\mathcal{M}, \tau, v_i, \vec{v}, \vec{\mu} \models \varphi$.
- ($\uparrow_{\mathbf{M}}$) $\mathcal{M}, \tau, x, \vec{v}, \vec{\mu} \models \uparrow_{\mathbf{M}}^i \varphi$ iff $\mathcal{M}, \tau, x, \vec{v}, \vec{\mu}_{[\tau/\mu_i]} \models \varphi$.
- ($\downarrow_{\mathbf{M}}$) $\mathcal{M}, \tau, x, \vec{v}, \vec{\mu} \models \downarrow_{\mathbf{M}}^i \varphi$ iff $\mathcal{M}, \mu_i, x, \vec{v}, \vec{\mu} \models \varphi$.

Whereas $\uparrow_{\mathbf{T}}^i \varphi$ replaces the i^{th} value of $\vec{v}$ with the evaluation time and $\uparrow_{\mathbf{M}}^i \varphi$ replaces the i^{th} value of $\vec{\mu}$ with the evaluation world, $\downarrow_{\mathbf{T}}^i \varphi$ shifts the evaluation time to the i^{th} value stored in $\vec{v}$ and $\downarrow_{\mathbf{M}}^i \varphi$ shifts the evaluation world to the i^{th} value stored in $\vec{\mu}$. These operators generalize the *now* and *then* operators that Vlach [35] introduces to resolve tense anaphora, replacing Vlach's single pair of operators with an indexed family $\uparrow_{\mathbf{T}}^i$ and $\downarrow_{\mathbf{T}}^i$ as well as providing their world-theoretic counterparts $\uparrow_{\mathbf{M}}^i$ and $\downarrow_{\mathbf{M}}^i$ to store and recall arbitrarily many times and worlds. Despite adding $\vec{v}$ and $\vec{\mu}$ as parameters to

⁵²The *strict semantics* given here is analogous to the *irreflexive* semantics for $\Box$ and $\Diamond$. See footnote 40.

the point of evaluation, the semantics for $\mathcal{L}$ may otherwise be maintained, letting $\mathcal{L}^* := \langle \mathbb{L}, \perp, \rightarrow, \Box, \triangleleft, \triangleright, \Box, \uparrow_T^i, \downarrow_T^i, \uparrow_M^i, \downarrow_M^i \rangle$. Although extending **TM** to provide a logic for $\mathcal{L}^*$ is outside the scope of the present paper, the following section will employ the semantics for $\mathcal{L}^*$ to address the paradox of the open future and draw further connections to non-deterministic dynamical systems theory.

4 Tense and Modality

The construction of possible worlds assumes that time is a total order. For any world $\tau \in H_{\mathcal{F}}$ and time $x \in D$ in a model $\mathcal{M}$ of $\mathcal{L}$, there is a determinate past and future relative to x where every well-formed sentence of $\mathcal{L}$ has a unique truth-value in τ at x . However natural it may be to take the past to be determined up to any given moment, it is much more contentious to suppose that each time determines a unique future. Arguments as old as Aristotle have sought to lend credence to the idea that the future differs from the past on account of remaining open to determination by admitting a range of incompatible future alternatives, none of which is singled out as the actual future. Thomason [25] brings this point out as follows:

[T]he basic issue here seems to be whether or not one is prepared to accept as meaningful the assertion that there is always, whether we know it or not, a single possible future which, from the perspective of a given time will be its actual future. (p. 270)

In considering future contingents such as ‘There will be a sea battle tomorrow’, many have sought to follow Aristotle in denying that such claims have a truth-value until the future comes about. Although there may be no fact of the matter whether there will be a sea battle tomorrow when it is considered today, we may nevertheless expect it to be settled tomorrow whether there ends up being a sea battle or not.

The construction of possible worlds takes each time in each possible world to have a complete past and future that is determined in every respect. However, evaluating a future contingent claim at a possible world and time does not carry any commitment that the world of evaluation is the actual world whose future is guaranteed to take place. By considering all worlds that intersect the world of evaluation at the time of evaluation, one may leave it open which of those worlds will play out. Just as one may consider the various chess games that may transpire from a given board state, it is also natural to consider the various routes between two points on a map, the different states a computer may transition through, or the different sequences of actions by one or more agents. Despite making evaluations according to one possible world or another when considering the evolution of a system, doing so does not require any of the possible worlds to be the *actual* world that will take place. Even if none of the possible worlds are foretold to be actual, we can say in full confidence what each possible world specifies. The following subsection will defend an approach of this kind by considering the shortcomings that face the alternatives.

4.1 Open Future

Dynamical systems of all kinds admit an open future. Chess continues to serve as a paradigm case. If each board state determined all future board states, there would only

be one game of chess to rehearse, leaving little semblance of a game worth playing. Just as the moves in chess narrow the range of open futures by determining what future board state becomes actual, human actions within non-deterministic systems carry similar consequences. When I choose soup over salad, I determine the next course of my meal, ruling out the futures in which salad precedes the main course while leaving open futures in which I go on to have dessert as well as those in which I do not.

By formally describing both deterministic and non-deterministic systems, we may ask as a strictly empirical matter whether any given system is deterministic or not. Whereas dynamical systems theory provides resources for modeling both deterministic and non-deterministic systems, **TM** provides logical resources for reasoning about dynamical systems. In particular, consider the following axiom schema:

$$\textit{Determined: } \varphi \rightarrow \Box\varphi.$$

If we happen to inhabit a deterministic universe, then all instances of the schema above are true. For instance, if I am going to visit Ladakh, then I will eventually visit Ladakh, or in symbols: $\Diamond p_l \rightarrow \Box p_l$. More generally, consider the frame constraint:

$$\text{DETERMINISTIC: For } w, u, v \in W \text{ and } x \in D, \text{ if } w \Rightarrow_x u \text{ and } w \Rightarrow_x v, \text{ then } u = v.$$

Whereas the frame constraints given in §3.3 impose a restriction on $\mathcal{D}$, the constraint above articulates what it is for the task relation $\Rightarrow$ to be deterministic. Accordingly, **T11** shows that *Determined* is valid over every deterministic frame, though (unlike the $\mathcal{D}$ -constraints above) not only over deterministic frames.

The present approach distinguishes between deterministic and non-deterministic systems independent of temporal structure. Thus **TL** may be retained, maintaining compatibility with the view that it is analytic that time is linear. Within each possible world $\tau \in H_{\mathcal{F}}$, time is a total order where every sentence receives a determinate truth-value at each time in τ . The openness of the future consists not in the existence of incomparable future times but in the absence of any possible world which is designated as the *actual* world: different possible worlds may pass through the same world state, where none is singled out as the one that has and will continue to obtain. This contrasts with the more standard proposal to accommodate an open future by dropping **TL** and replacing the linear temporal order with a partial order. The remainder of this section considers the difficulties that branching-time faces and explains how the task semantics achieves a superior result while retaining a linear theory of time.

Instead of requiring the times in each frame to be totally ordered as in §3, there is the tradition following Prior [21] and Kripke [20] that weakens the tense logic by dropping **TL** and taking the times to form a connected left-linear partial order so that each time $x \in D$ may admit incomparable futures.⁵³ Despite admitting branching times, one might nevertheless preserve the definition from §3 that a world history is any function $\tau : X \rightarrow W$ where $X \subseteq D$ is a nonempty convex set of times and $\tau(x) \Rightarrow_{y-x} \tau(y)$ for all $x, y \in X$, referring to world histories as *possible worlds*. This proposal faces the same difficulties that arise for the Peircean semantics considered in

⁵³One must also give up **TD** to validate $\Diamond\top \rightarrow (\Box\Box\varphi \rightarrow \Delta\varphi)$ which requires the past to be unique and then add the result of exchanging $\Box$ and $\Diamond$ to recover what is otherwise derivable from **TD**.

§2.1. For instance, consider a chess game τ at a time x in which there is a future time $y > x$ where the black king is in checkmate and an incomparable future time $z > x$ where the white king is in checkmate. By the semantics for the future operator $\Diamond$, ‘White is going to win’ and ‘Black is going to win’ are both true, though this would seem impossible. It is much more natural to assert ‘White *could eventually* win’ and ‘Black *could eventually* win’ to indicate that there is some future in which White wins and some future in which Black wins, where these are the sorts of chess games that are still worth playing. Such a theorist might take $\Diamond$ to read ‘could eventually’ which requires φ at some later time in some branch, and take $\Box$ to read ‘will always’ which requires φ at all later times across all branches. Although this reading is more fitting here, a branching-time theory of this kind inherits the expressive limitations of the Peircean semantics. For instance, there is no way to express the density of time, insofar as density $\Box\Box\varphi \rightarrow \Box\varphi$ concerns the structure of any one branch rather than all future times distributed over all branches. Following the Ockhamist by introducing a further parameter at which to evaluate sentences that specifies the particular branch within a possible world returns the situation that we were in before where sentences are evaluated at parameters that suffice to determine a fixed past and future.

Consider a game of chess in which both Black and White have a chance at winning after 14 moves have been played. Given the state of the board, there simply is no actual game with a determinate future that proceeds from the present board state. Rather, we may consider various chess games that proceed from the present board state at move 14. By taking turns choosing which moves to make, the players compete in their abilities to determine which game of chess becomes actual. Rather than building indeterminacy into the temporal structure itself by taking time to be only partially ordered, the task semantics for $\mathcal{L}$ locates indeterminacy in the task relation between world states over a duration. Letting time be a total order, each possible world $\tau \in H_{\mathcal{F}}$ represents one possible path through the space of world states, constrained only by the transitions which the task relation permits. Different possible worlds intersecting the same world state represent genuine alternatives about how the system could evolve without requiring multiple incomparable future times. So long as none of these possible worlds are assumed to be actual, no harm comes in allowing each possible world to specify a determinate past and future from any given world state at a time.

Something similar may be said for Aristotle’s sea battle. Since some of the possible worlds intersecting the present world state include a sea battle on the following day and other intersecting possible worlds do not, it is contingent whether there is going to be a sea battle. In symbols, one might assert $\Diamond\neg p_s \wedge \Diamond p_s$ where p_s expresses that a sea battle is taking place and $\Diamond\varphi := \Diamond\Diamond\varphi$ as before, though this is easy to satisfy.⁵⁴ Given a possible world in a deterministic frame where the ships first clash at 2pm tomorrow, both $\Diamond\neg p_s$ and $\Diamond p_s$ could be true as witnessed by the time 11am while the ships were still yet to meet and 2:30pm just after the battle started. Rather, we may consider:

$$\uparrow_T^1 \Diamond \uparrow_T^2 \downarrow_T^1 (\Diamond \downarrow_T^2 \neg p_s \wedge \Diamond \downarrow_T^2 p_s) \quad (\text{Sea})$$

⁵⁴Given that $\Diamond\neg p_s \wedge \Diamond p_s$ is satisfiable, $\Box\varphi \vee \Box\neg\varphi$ is invalid despite retaining **TL** and **TD**. However, invalidity is had too easily, since we need only consider a single world in which there are future times where φ and other future times where $\neg\varphi$. Something similar may be said for the satisfiability of $\Diamond\neg p_s \wedge \Diamond p_s$.

The claim above asserts that there is a future time where p_s is contingent on account of being true in one possible future and false in another. More generally, we may consider the following schema which is invalid given the truth of the claim above:

$$\uparrow_T^1 \Box \uparrow_T^2 \downarrow_T^1 (\Box \downarrow_T^2 \neg \varphi \vee \Box \downarrow_T^2 \varphi) \quad (\text{Det})$$

For a given future time, the possible worlds intersecting the present need not agree on φ at that time. In particular, during a day of strategic uncertainty, it may be indeterminate whether there will be a sea battle tomorrow on account of possible worlds that intersect with the present in which there is a sea battle twenty-four hours from now as well as intersecting possible worlds in which there is not.⁵⁵

As **T12** shows, **Det** is valid over any deterministic frame. By contrast, it is easy to conceive of non-deterministic systems in which **Det** is false where **Sea** requires this to be the case. Given a possible world $\tau \in H_{\mathcal{F}}$, time $x \in D$, and stored times $\vec{v}$ and worlds $\vec{\mu}$ in a model $\mathcal{M} = \langle W, \mathcal{D}, \Rightarrow, |\cdot| \rangle$ where **Sea** is true, there are at least two possible worlds $\sigma_1, \sigma_2 \in \langle \tau \rangle_x$ where there is a sea battle in σ_1 at time y and there is no sea battle in σ_2 at time y . Considered from the world state $\tau(x) = \sigma_1(x) = \sigma_2(x)$, it would be wrong to claim that σ_1 is the actual world that is destined to be, just as it would be wrong to claim that σ_2 is the actual world. However, when evaluated from $\sigma_1(y)$, there is good reason to take σ_1 to have a clear advantage over σ_2 , and *vice versa* when evaluated from $\sigma_2(y)$. After all, a sea battle occurs in σ_1 at time y but not so in σ_2 , where this is a major difference that cannot be ignored. Even if σ_1 has a better claim to being actual when we arrive at $\sigma_1(y)$, one cannot claim that σ_1 was actual at time x given the intersecting possible world σ_2 and perhaps many others.

The paradox of the open future may be stated as follows: (1) there is a unique actual world; (2) if there is a unique actual world, then there is no future contingency; (3) it appears that there is future contingency. Articulated in terms of the task semantics, the solution is that there is no *actual world* $\alpha \in H_{\mathcal{F}}$ with a determinate past and future. If there were an actual world, one might include it among the semantic primitives in a frame $\mathcal{F}_{@} = \langle W, \mathcal{D}, \Rightarrow, \alpha \rangle$ where α is taken to at least simulate actuality to provide a semantics for an actuality operator $@$ as follows:

$$(@) \mathcal{M}, \tau, x, \vec{v}, \vec{\mu} \models @\varphi \text{ iff } \mathcal{M}, \alpha, x, \vec{v}, \vec{\mu} \models \varphi.$$

Including an actual world α in a frame raises questions about the commitments of the intended models for the language. By contrast, $\uparrow_M$ and $\downarrow_M$ store and recall worlds without any reference to a globally actual world, avoiding the apparent commitment to an actual future while still providing expressive resources for cross referencing worlds. Instead of positing an actual world α with a complete past and future, a weaker commitment takes each frame $\mathcal{F}_{\#} = \langle W, \mathcal{D}, \Rightarrow, \# \rangle$ to include a distinguished *present* $\# \in W$ representing the world state that currently obtains.⁵⁶ Given $\#$ and fixing a time

⁵⁵Carinai and Santorio [36] treat ‘will’ as a non-quantificational modal governed by a selection function rather than a universal quantifier over worlds. To retain bivalence and reject MacFarlane’s [37] relativized truth, the indeterminacy of future contingents is located in facts about a plurality of live alternatives rather than in truth-value gaps. The present proposal is more deflationary still, since no possible world is designated actual, even indeterminately.

⁵⁶Alternatively, one might designate a *past* that includes the world states that have obtained up to and including the world state that currently obtains, providing a growing block theory of time with Broad [38].

n , we may consider all possible worlds that occupy $\#$ at n without any commitment to there being a world that includes an actual future. Although far less committing than $\mathcal{F}_@$, the significance of a frame such as $\mathcal{F}_\#$ is fleeting since what obtains now is only momentary. In contrast to both of these proposals, no world state $\# \in W$ has been designated as present, and no possible world $@ \in H_{\mathcal{F}}$ has been designated as the actual world. Instead of positing the structure required to say what is true *simpliciter*, the semantics for $\mathcal{L}$ only includes what is needed to evaluate truth relative to one world and time or another where none are distinguished.

Looking out at the Aegean, Aristotle may have wondered whether there will be a sea battle tomorrow or not. Although there are various possible worlds intersecting with the present according to which there is going to be a sea battle tomorrow as well as possible worlds in which there is not, no possible world is designated as the actual world that is already destined to take place. Rather, it remains open which trajectory through the space of world states we may traverse where the future that becomes actual is often determined in part by our influence. By the time tomorrow comes we will know whether there has been a sea battle or not, and so nothing regarding the sea battle will remain open to determination. But arriving where we have by one trajectory or another does not entail that we were always bound to arrive where we have from any point in the past. That we may model our various possible trajectories through the space of world states does not require any trajectory to be actual. At most we may say that the present obtains, though the present is fleeting.

4.2 Dynamical Systems

Dynamical systems theory provides a general mathematical framework for modeling the possible evolutions of a system, suggesting a natural connection with tense and modal reasoning. Whereas the frames defined above take time to have both group and order structure, non-deterministic dynamical systems typically consist of a set of states W , a monoid $\langle D, +, 0 \rangle$ for positive durations, and a set of relations $\{R_x\}_{x \in D}$ indexed by D that satisfy the following conditions for all $w, u, v \in W$ and $x, y \in D$:⁵⁷

Nullity: $R_0(w, w)$.

Compositionality: If $R_x(w, u)$ and $R_y(u, v)$, then $R_{x+y}(w, v)$.

Amounting to no more than a change of notation, $R_x(w, u)$ expresses with an indexed family of two-place relations what $w \Rightarrow_x u$ expresses with a three-place relation. However, to provide semantic clauses for tense operators which quantify over all durations that are *less than* or *greater than* a given duration of evaluation, an order must also be provided. Since it is just as natural to subtract durations as it is to add them where this is used to state *Reflection*, I will assume that every duration x has an inverse $-x$, writing $x - y$ in place of $x + (-y)$ as usual. Moreover, the order of addition does not make a difference, motivating the commutativity $x + y = y + x$ constraint. Thus I have restricted attention to dynamical systems in which the durations $\mathcal{D} = \langle D, +, 0, \leq \rangle$ form a totally ordered abelian group. Given that the task relation induces a canonical

⁵⁷A *monoid* is any $\langle M, \cdot, 1_M \rangle$ where: (1) $a \cdot b \in M$ whenever $a, b \in M$; (2) $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ for all $a, b, c \in M$; and (3) $a \cdot 1_M = 1_M \cdot a = a$ for all $a \in M$.

R_0 topology on world states, **T13** establishes that task frames are always R_0 , where **T14** shows that this topology is in general non-discrete.

A dynamical system is *deterministic* if R_x is a total function for all $x \in D$, justifying the notation for the *world functions* $\{f_x\}_{x \in D}$ where $f_x(w) = u$ just in case $R_x(w, u)$. A *deterministic model* is any $\mathcal{M}_D = \langle W, \mathcal{D}, \{f_x\}_{x \in D}, |\cdot| \rangle$ where $|p_i| \subseteq W$ for all $p_i \in \mathbb{L}$ and $\langle W, \mathcal{D}, \{f_x\}_{x \in D} \rangle$ is the deterministic dynamical system. Drawing on these resources, Williamson [4, 5] provides a semantics for a higher-order bimodal language in terms of the primitive world functions provided by a deterministic model. I will present the propositional fragment of Williamson’s semantics as follows:

- $(p_i) \mathcal{M}_D, w \models p_i \text{ iff } w \in |p_i|.$
- $(\perp) \mathcal{M}_D, w \not\models \perp.$
- $(\rightarrow) \mathcal{M}_D, w \models \varphi \rightarrow \psi \text{ iff } \mathcal{M}_D, w \not\models \varphi \text{ or } \mathcal{M}_D, w \models \psi.$
- $(\Box) \mathcal{M}_D, w \models \Box \varphi \text{ iff } \mathcal{M}_D, u \models \varphi \text{ for all } u \in W.$
- $(\Box) \mathcal{M}_D, w \models \Box \varphi \text{ iff } \mathcal{M}_D, f_x(w) \models \varphi \text{ for all } x \in D \text{ where } x < 0.$
- $(\Box) \mathcal{M}_D, w \models \Box \varphi \text{ iff } \mathcal{M}_D, f_x(w) \models \varphi \text{ for all } x \in D \text{ where } x > 0.$

Instead of interpreting sentences at a model, possible world, and time, the semantic clauses above evaluate sentences directly at a model and world state. This approach may claim the advantage of validating the perpetuity principles **P1** – **P6** from before without undermining the intelligibility of the truth-conditions for the sentences of the language. Nevertheless, the semantics above is restricted to deterministic systems, and so unable to model systems in which states fail to determine a unique past and future. For instance, the board states in a game of chess do not typically mandate any particular continuation, leaving open many different incompatible futures.

Since world functions do not accommodate any degree of future contingency, one might attempt to weaken the temporal order in an attempt to encode an open future. As already observed, taking $\mathcal{D}$ to be a partial order presents the difficulty facing the Peircean semantics which quantifies over all or some incomparable future times. For instance, it is absurd to admit that both Black and White are going to win given incomparable future times in which Black and White each win. Instead of attempting to accommodate the open future, it is natural to restrict applications of Williamson’s semantics to deterministic systems where exactly one history intersects each world state. For instance, the motions of the planets are well modeled by a deterministic system constrained under Newton’s laws. Given the total state of the solar system including the position and momentum of each planet, all future states of that system are fully determined. If world states determine their past and future, there is no need to specify a temporal parameter alongside a possible world, obviating the need to construct possible worlds. Even so, these are small gains in compensation for giving up the ability to represent non-deterministic systems, especially when metaphysical modality is under discussion. If there are systems whose world states have incompatible futures, the semantics for the strongest objective modality must quantify over all futures for those systems rather than requiring the future to be unique.

The study of dynamical systems is not new. Since the time of Galileo, Newton, and Leibniz during the seventeenth century, it has been standard practice to represent the evolution of a system by using functions. By contrast, it is unheard of not just in physics but throughout the sciences to represent the evolution of a system by a primitive point. The dynamics described by the semantics for languages with tense and modal operators are no different. Thomason [39] presses this point as follows:

Physics should have helped us to realize that a temporal theory of a phenomenon X is, in general, more than a simple combination of two components: the statics of X and the ordered set of temporal instants. The case in which all functions from times to world states are allowed is uninteresting; there are too many such functions, and the theory has not begun until we have begun to restrict them. [...] The general moral, then, is that we shouldn't expect the theory of time + X to be obtained by mechanically combining the theory of time and the theory of X . (p. 135)

Montague [1] and Kaplan's [2] two-dimensional semantics assumes a dynamics that predates the development of functional dynamics during the Scientific Revolution. Although dynamical systems are nothing new, what has not been adequately explored is the connection between functional dynamics and bimodal reasoning. This paper provides a step in that direction. In the following section, I will conclude by drawing connections to a number of recent developments in logic and computer science.

4.3 Conclusion

Defining possible worlds to be task-constrained functions $\tau : X \rightarrow W$ from a convex set of durations to world states validates the perpetuity principles **P1** – **P6** while maintaining an expressive semantics for the tense and modal operators. By encoding the full range of non-deterministic dynamical systems, the task semantics accommodates an open future without weakening the temporal order: indeterminacy is located in the diversity of possible worlds passing through a given world state rather than branching time. The soundness **T20** and completeness **C3** theorems show that **TM** provides resources that are both correct and adequate for reasoning about non-deterministic dynamical systems, where the perpetuity principles are derived from the sole bimodal interaction axiom **MF** which axiomatizes the free translation of possible worlds.

These results address the deficiencies of extant bimodal logics. Whereas Montague [1] works with a less expressive modality that trivializes the perpetuity principles, Kaplan's [2] semantics invalidates perpetuity. Restricting to the abundant models in §2.3 restores validity, but only by imposing a model constraint in place of a frame constraint, forfeiting the correspondence between **MF** and the temporal structure. It is this valuation-dependence that forces abundance theorists to posit all merely temporal differences between primitive worlds, and so to embrace temporal absolutism or instrumentalism. Williamson's [4] deterministic semantics validates the perpetuity principles but is restricted to systems in which each world state fixes a unique future, and so is unable to represent non-deterministic systems. By introducing a task relation to constrain transitions between world states over a duration without requiring the task relation to be functional, the task semantics validates the perpetuity principles, handles non-determinism, avoids temporal absolutism, and preserves the expressive power of the semantic clauses for the tense and modal operators.

The task semantics is closely related to several neighboring research programs. Belnap, Perloff, and Xu’s [40] agentic STIT theory and Rumberg’s [27] transition semantics follow Thomason’s [25] reconstruction of Prior’s [21] Ockhamist semantics. Both frameworks represent possibilities without committing to a fully determined actual future by taking the temporal order to be partial, admitting incomparable moments that branch into alternative futures. By contrast, the task semantics takes the durations $\mathcal{D}$ to be totally ordered, accommodating indeterminism by way of the task relation which unfolds into a branching tree instead of positing the tree outright. Instead of defining the models over a partial order $\langle T, < \rangle$, the models of $\mathcal{L}$ are defined over a task frame $\mathcal{F} = \langle W, \mathcal{D}, \Rightarrow \rangle$ which is a labeled transition system (LTS) where the transitions are provided by the task relation $w \Rightarrow_x u$ between world states and the labels are durations in $\mathcal{D}$ carrying the group structure.⁵⁸ Since the domain $X \subseteq \mathcal{D}$ for each possible world is linearly ordered, the framework avoids the difficulties that beset the Peircean future tense operator over branching time outlined in §2.1. When W is finite and $D = \mathbb{Z}$, the set of complete possible worlds $H_{\mathcal{F}}^*$ is a shift of finite type: world states correspond to alphabet symbols, the unit-step task relation corresponds to the edges of a labeled graph, and complete possible worlds correspond to the bi-infinite paths through that graph.⁵⁹ The general construction extends this correspondence to continuous-time and infinite-state systems.⁶⁰ A possible world is a labeled run of this dynamical system—a function $\tau : X \rightarrow W$ from times to world states rather than an order over world states made to serve as times. Representing the evolution of a system by functions rather than primitive points is the moral that Thomason [39] himself presses in the passage quoted in §4.2. The first-order definability and axiomatizability results that Rumberg and Zanardo [45] establish provide a useful benchmark that the present framework meets: **T8**, **T9**, and **T10** give the corresponding definability results where the completeness result **C3** extends to each class of frames.

The debate in computer science between branching-time and linear-time temporal logics receives a distinctive resolution in the present framework.⁶¹ Individual possible worlds are linearly ordered as in LTL [49], while the set of possible worlds through a given world state has a branching structure as in CTL [50]. The key difference is that $\Box$ quantifies universally over all possible worlds in $H_{\mathcal{F}}$ at the same time rather

⁵⁸A labeled transition system is a triple $\langle S, \Lambda, \rightarrow \rangle$ consisting of a set S of states, a set Λ of labels, and a transition relation $\rightarrow \subseteq S \times \Lambda \times S$, written $s \xrightarrow{a} s'$. The relation need not be functional, since a state may have several a -labeled successors or none, which is exactly what allows the task relation $\Rightarrow_x$ to carry indeterminism. See Baier and Katoen [41] for a comprehensive treatment.

⁵⁹A shift space X over a finite alphabet A is a *shift of finite type* if and only if there exists a finite directed graph $G = (V, E)$ with a labeling function $L : E \rightarrow A$ such that X is the set of all bi-infinite sequences obtained by reading the labels along bi-infinite paths in G . A *sofic shift* is the strictly more general class of shifts obtained as a factor of a shift of finite type under a sliding block code. See Lind and Marcus [42] for a comprehensive treatment.

⁶⁰Task frames are instances of coalgebras for a duration-indexed family of functors, and the properties of the task frame (*Nullity*, *Reflection*, *Compositionality*) endow the transition structure with the structure of a groupoid graded by $(D, +, 0)$ on W . The transition structure constitutes a groupoid $\mathbf{G}$ with object set W , in which a morphism from w to u of grade $x \in D$ exists precisely when $w \Rightarrow_x u$. The *Nullity*, *Reflection*, and *Compositionality* axioms are exactly the identity, inversion, and composition laws of a groupoid, together with a grading functor $\kappa : \mathbf{G} \rightarrow \mathbf{BD}$ to the one-object groupoid on $(D, +, 0)$. The non-functionality of $\Rightarrow_x$ distinguishes this from an ordinary group action, where each $x \in D$ would determine a unique function $W \rightarrow W$. See Weinstein [43] for a survey of groupoids and Rutten [44] for universal coalgebra.

⁶¹The debate was initiated by Lamport [46] and later given its “final showdown” by Vardi [47]. Emerson and Halpern’s CTL* [48] forms a superset of both LTL and CTL by combining path quantifiers with temporal operators freely rather than having either logic subsume the other.

than employing the state-relativized path quantifiers A and E of CTL. This universal quantification is what validates the perpetuity principles: $\Box\varphi \rightarrow \Delta\varphi$ holds because the time-shift invariance of $H_{\mathcal{F}}$ guarantees that every possible world satisfying φ at one time has a shifted counterpart satisfying φ at any other. Whereas **TM** validates its bimodal interaction principle by the construction of possible worlds itself, this fails in the broader landscape of products of modal logics.⁶² We may also restrict quantification to intersecting worlds with $\Box$, or to worlds with a common past and present relative to the world and time of evaluation by adding $\boxdot$. HyperLTL provides a parallel treatment by extending LTL with quantifiers $\forall\pi$ and $\exists\pi$ ranging over the execution traces of a system. Rather than a bimodal logic with tense and modal operators, HyperLTL is a tense logic with first-order quantifiers for traces, making satisfiability in HyperLTL undecidable. By contrast, the semantics for $\Box$ quantifies over the possible worlds in $H_{\mathcal{F}}$ without binding traces within the object language, making **TM** decidable **C4** by the finite model property and completeness **C3**.⁶³

The *Logic of Tense and Modality* **TM** provides resources for reasoning about non-deterministic dynamical systems, combining tense and modal operators to give expression to past and future contingency. Incorporating propositional quantifiers would make higher-order quantification over propositions (sets of world states) available in the object language, bringing the framework closer to the language that Williamson [4, 5] employs. The agentive choice function that Belnap, Perloff, and Xu [40] add to a branching theory of time partitions the intersecting histories at a moment into an agent’s available choices. Introducing such an overlay would define agentive operators over the non-agentive dynamics developed here.⁶⁴ Probabilistic extensions would connect the framework to stochastic dynamical systems. These directions remain to be elaborated elsewhere.

5 Appendix

The following sections present results to support the main body of the paper. In §5.1, I characterize $\Box$ as the strongest objective normal modal operator, proving it unique up to equivalence and deriving the **S4** axiom. Although I will not assume Booleanism, I establish existence under a coarse-grained strengthening of propositional identity in **C1** in order to draw a direct comparison with Bacon [57]. The two-dimensional results in §5.2 support the arguments in §2. The task semantics is developed independently in §5.3, where §5.4 provides elements of the soundness theorem. I then introduce the proof system **TM**⁺ in §5.5, proving it to be a conservative extension of **TM** in **T23**. The completeness of **TM** and its extensions is given by **C3**, where the decidability

⁶²See Kurucz [31] for a general treatment of the products of modal logics. The task relation is also structurally analogous to the program relation in the dynamic logic presented by Harel, Kozen, and Tiuryn [51], though the present framework constructs possible worlds rather than modeling program executions, and its labels carry abelian group structure that dynamic logic does not require.

⁶³Clarkson et al. [52] introduced HyperLTL, and Finkbeiner and Zimmermann [53] showed that it translates into a proper fragment of first-order logic with an equal-level predicate relating positions at the same time on distinct traces. Finkbeiner and Hahn [54] proved that its satisfiability problem is undecidable, whereas Finkbeiner, Rabe, and Sánchez [55] showed that model checking against a given finite-state system is decidable, with complexity rising by an exponential for each quantifier alternation.

⁶⁴See Horty [56] for a STIT-based treatment of agentive operators.

of **TM** in **C4** follows from the finite model property. All results supporting the task semantics are implemented in Lean 4 in the repository for this paper.⁶⁵

5.1 Objective Modality

This section characterizes metaphysical necessity as the *strongest objective normal modal operator*, deriving elements of its logic from that characterization. The base language $\mathcal{L}$ is extended with a primitive propositional identity operator together with higher-order quantifiers. Operator variables range over an unrestricted domain of operations on propositions. The treatment is predicative: operator comprehension is confined to formulas containing no operator variables. This keeps the system consistent with a fine-grained $\equiv$, blocking the Russell–Myhill paradox.⁶⁶

D1 An *identity extension* of $\mathcal{L}$ is a language $\mathcal{L}^\equiv$ that enriches $\mathcal{L}$ with a binary propositional identity operator $\equiv$, read ‘For φ to be true is for ψ to be true’, whose logic comprises classical propositional logic and the minimal theory of identity given below, where $\chi_{(\psi/\varphi)}$ is the result of replacing one or more occurrences of φ in any formula χ with ψ :

$$\begin{array}{ll} \mathbf{Ref} & \vdash \varphi \equiv \varphi. \\ \mathbf{LL} & \vdash (\varphi \equiv \psi) \rightarrow (\chi \rightarrow \chi_{(\psi/\varphi)}). \\ \mathbf{Imp} & \vdash (\varphi \equiv \psi) \rightarrow (\varphi \rightarrow \psi). \end{array}$$

The theory of propositional identity need not be Boolean, accommodating theories in which the absorption laws or other Boolean identities do not hold.⁶⁷ Symmetry and transitivity of $\equiv$ are nevertheless derivable: given $\varphi \equiv \psi$, applying **LL** to $\chi := (\varphi \equiv \varphi)$ yields $\psi \equiv \varphi$ by **Ref**, and a further application gives transitivity.

D2 The *quantified identity extension* $\mathcal{L}_2^\equiv$ of $\mathcal{L}^\equiv$ adds a countable stock of one-place operator variables $\mathcal{Q}, \mathcal{P}, \dots$, propositional quantifiers $\forall p$ binding the sentence letters, and operator quantifiers $\forall \mathcal{Q}$ binding the operator variables. The *operator terms* are generated from the operator variables, the trivial operator $\mathbf{I}$, and the modal primitive $\Box$ under composition $\circ$:

$$T, T' ::= \mathbf{I} \mid \Box \mid \mathcal{Q} \mid T \circ T',$$

and the *well-formed formulas* are defined recursively:

$$\varphi, \psi ::= p_i \mid \perp \mid \varphi \rightarrow \psi \mid \Box \varphi \mid \mathbb{P} \varphi \mid \mathbb{F} \varphi \mid \varphi \equiv \psi \mid T \varphi \mid \forall p \varphi \mid \forall \mathcal{Q} \varphi.$$

Since each operator term T applies to a single formula, every operator is one-place and the operator quantifier $\forall \mathcal{Q}$ binds one-place operator variables. The grammar clause $\Box \varphi$ is thus subsumed by $T \varphi$.

⁶⁵See <https://github.com/benbrastmckie/BimodalLogic> for the Lean 4 repository for this paper.

⁶⁶Predicativity is a standard response to Russell–Myhill. See Walsh [58] for an explicit consistency proof of a predicative restriction of comprehension in Church’s intensional logic. Predicativity could be dropped by strengthening the theory of $\equiv$ as **C1** shows, since coarse-grained identity blocks Russell–Myhill independently.

⁶⁷I defend a bilateral theory of propositional identity in [59].

D3 A logic L for $\mathcal{L}_2^=$ extends the logic from **D1** to include the axioms and rules below, where $\varphi[\psi/p]$ replaces *every* free occurrence of p by ψ with ψ free for p , and $\varphi[T/Q]$ replaces every free occurrence of Q by the operator term T with T free for Q , and where **LL** is henceforth restricted so that no variable free in φ or ψ is bound at a replaced occurrence in χ :

1. $\vdash I\varphi \equiv \varphi$
2. $\vdash \forall p \varphi \rightarrow \varphi[\psi/p]$
3. $\vdash (T \circ T')\varphi \equiv T(T'\varphi)$
4. $\vdash \forall Q \varphi \rightarrow \varphi[T/Q]$
5. $\vdash \varphi$ implies both $\vdash \forall p \varphi$ and $\vdash \forall Q \varphi$
6. $\vdash \forall p(\varphi \rightarrow \psi) \rightarrow (\varphi \rightarrow \forall p \psi)$ provided the variable p is not free in φ
7. $\vdash \forall Q(\varphi \rightarrow \psi) \rightarrow (\varphi \rightarrow \forall Q \psi)$ provided the variable Q is not free in φ
8. $\vdash \exists Q \forall p (Qp \equiv \theta)$ for θ containing no operator variables and at most p free (where $\exists$ abbreviates $\neg \forall \neg$).⁶⁸

T1 $\vdash \forall Q \text{Tra}(Q)$ where $\text{Tra}(Q) := \forall p \forall q [(p \equiv q) \rightarrow (Qp \equiv Qq)]$.

Proof. Applying **LL** to $\chi := (Qp \equiv Qp)$, replacing the second occurrence of p with q , yields $(p \equiv q) \rightarrow [(Qp \equiv Qp) \rightarrow (Qp \equiv Qq)]$, where the antecedent $Qp \equiv Qp$ is discharged by **Ref**, leaving $(p \equiv q) \rightarrow (Qp \equiv Qq)$. Generalizing on p , q , and Q gives $\forall Q \text{Tra}(Q)$. $\square$

D4 $\text{Obj}(Q) := \text{Tra}(Q) \wedge \text{Fac}(Q)$ where $\text{Fac}(Q) := \forall p (Qp \rightarrow p)$.

D5 $Q \leq P := \forall p (Qp \rightarrow Pp)$ which reads ‘ Q is at least as strong as P ’.

D6 $\text{Ax}(Q) := Q\top \wedge \forall p \forall q [Q(p \rightarrow q) \rightarrow (Qp \rightarrow Qq)]$ where $\top := (\perp \rightarrow \perp)$.⁶⁹

D7 Q is a *normal modal operator in L* — $\text{Norm}_L(Q)$ — iff:

1. $\vdash \text{Ax}(Q)$; and
2. L is closed under Q -necessitation: $\vdash \varphi$ implies $\vdash Q\varphi$.⁷⁰

D8 Q is a *strongest objective normal modal operator in L* — $\text{Str}_L(Q)$ — iff:

1. $\vdash \text{Obj}(Q)$;
2. $\text{Norm}_L(Q)$; and
3. $\vdash \forall P [(\text{Obj}(P) \wedge \text{Ax}(P)) \rightarrow (Q \leq P)]$.

⁶⁸See again Walsh [58] for the consistency of restrictions of this kind.

⁶⁹The conjuncts $Q\top$ and the distribution schema are the modal axioms **N** and **K**.

⁷⁰The necessitation rule is not equivalent to any formula of $\mathcal{L}_2^=$: it concerns theoremhood, which not all theories of propositional identity $\equiv$ can internalize, since $\varphi \equiv \top$ may be strictly stronger than the validity of φ . Normality is therefore a condition on an operator in L , not a conjunct of the object-language predicate Obj ; below it is invoked only for the strongest operator itself, never for the quantified P , so it does not clash with the unrestricted operator domain or with comprehension.

L1 If $\text{Str}_L(\mathcal{Q})$ and $\text{Str}_L(\mathcal{P})$, then $\vdash \forall p(\mathcal{Q}p \leftrightarrow \mathcal{P}p)$.

Proof. Since $\vdash \text{Obj}(\mathcal{P})$ by clause (1) and $\mathcal{P}$ is normal, $\vdash \text{Ax}(\mathcal{P})$, so instantiating the dominance clause for $\mathcal{Q}$ at $\mathcal{P}$ yields $\vdash \forall p(\mathcal{Q}p \rightarrow \mathcal{P}p)$. Parity of reasoning yields $\vdash \forall p(\mathcal{P}p \rightarrow \mathcal{Q}p)$. $\square$

L2 If $\text{Norm}_L(\mathcal{Q})$, then $\vdash \text{Ax}(\mathcal{Q} \circ \mathcal{Q})$.

Proof. By the composition axiom it suffices to establish each conjunct for $\mathcal{Q}\mathcal{Q}$. Given $\vdash \mathcal{Q}\top$, we know $\vdash \mathcal{Q}\mathcal{Q}\top$ by necessitation for $\mathcal{Q}$ since $\mathcal{Q}$ is a normal modal operator. For distribution, let $\theta := \mathcal{Q}(p \rightarrow q) \rightarrow (\mathcal{Q}p \rightarrow \mathcal{Q}q)$ which is itself an instance of distribution for $\mathcal{Q}$, so $\vdash \theta$ and hence $\vdash \mathcal{Q}\theta$ by necessitation for $\mathcal{Q}$. Applying distribution for $\mathcal{Q}$ to θ gives $\vdash \mathcal{Q}\theta \rightarrow [\mathcal{Q}\mathcal{Q}(p \rightarrow q) \rightarrow \mathcal{Q}(\mathcal{Q}p \rightarrow \mathcal{Q}q)]$, so $\vdash \mathcal{Q}\mathcal{Q}(p \rightarrow q) \rightarrow \mathcal{Q}(\mathcal{Q}p \rightarrow \mathcal{Q}q)$. Distribution also has the instance $\vdash \mathcal{Q}(\mathcal{Q}p \rightarrow \mathcal{Q}q) \rightarrow (\mathcal{Q}\mathcal{Q}p \rightarrow \mathcal{Q}\mathcal{Q}q)$. Chaining these two implications yields $\vdash \mathcal{Q}\mathcal{Q}(p \rightarrow q) \rightarrow (\mathcal{Q}\mathcal{Q}p \rightarrow \mathcal{Q}\mathcal{Q}q)$, as required. $\square$

T2 If $\text{Str}_L(\mathcal{Q})$, then $\vdash \forall p(\mathcal{Q}p \rightarrow \mathcal{Q}\mathcal{Q}p)$.

Proof. By **L2**, $\vdash \text{Ax}(\mathcal{Q} \circ \mathcal{Q})$. Moreover $\vdash \text{Obj}(\mathcal{Q} \circ \mathcal{Q})$: transparency is an instance of **T1**, and factivity follows since $(\mathcal{Q} \circ \mathcal{Q})p \equiv \mathcal{Q}\mathcal{Q}p$ by composition, $\mathcal{Q}\mathcal{Q}p \rightarrow \mathcal{Q}p$ by instantiating $\text{Fac}(\mathcal{Q})$ at $\mathcal{Q}p$, and $\mathcal{Q}p \rightarrow p$. Instantiating the dominance clause for $\mathcal{Q}$ at the operator term $\mathcal{Q} \circ \mathcal{Q}$ yields $\vdash \text{Ax}(\mathcal{Q} \circ \mathcal{Q}) \rightarrow \forall p(\mathcal{Q}p \rightarrow (\mathcal{Q} \circ \mathcal{Q})p)$, and so $\vdash \forall p(\mathcal{Q}p \rightarrow \mathcal{Q}\mathcal{Q}p)$ by the composition axiom. $\square$

It is a substantive issue whether L includes a strongest objective normal modal operator. Since $\vdash \text{Ax}(\mathbf{l})$ and L is closed under $\mathbf{l}$ -necessitation given that $\vdash \mathbf{l}\varphi \equiv \varphi$, the trivial operator $\mathbf{l}$ is normal, though there need not be a $\leq$ -strongest objective normal modal operator. There is, however, a canonical candidate $\Box\varphi := (\varphi \equiv \top)$. Applying comprehension to the predicative formula $p \equiv \top$ guarantees an operator satisfying $\forall p(\mathcal{Q}p \equiv \Box p)$. This operator is objective: transparency is an instance of **T1**, and factivity holds since $p \equiv \top$ yields $\top \equiv p$ by symmetry, whence p by **Imp**. Moreover, $\vdash \Box\top$ by **Ref**, and $\vdash \forall \mathcal{P}[\text{Ax}(\mathcal{P}) \rightarrow (\Box \leq \mathcal{P})]$ since $\text{Ax}(\mathcal{P})$ yields $\mathcal{P}\top$ and **LL** carries $\mathcal{P}\top$ to $\mathcal{P}p$ given $\top \equiv p$. Thus $\Box$ satisfies every condition of **D8** except normality, which depends on L . Indeed, in fine-grained settings it is precisely the normality of $\Box$ — and hence existence— that fails. See Bacon [57, §3 and fn. 28] for related discussion.

D9 A logic L for $\mathcal{L}_2^\equiv$ includes the *Rule of Equivalence* just in case $\vdash \varphi \leftrightarrow \psi$ implies $\vdash \varphi \equiv \psi$, so that interderivable formulas are identical (Bacon [57, p. 746]).

C1 If L includes the *Rule of Equivalence* **D9**, then L includes a strongest objective normal modal operator, unique up to equivalence and given by $\Box$.⁷¹

⁷¹Bacon [57, p. 747] shows that **D9** is stronger than *Booleanism*— that Boolean-equivalent formulas are identical— which yields distribution but not necessitation. This corollary transposes Bacon’s results into the present setting for comparison: $\Box$ is the Cresswell–Suszko operator, for which Bacon proves **S4** (Prop. 3.1), that it is the broadest necessity (Prop. 4.1), and that it is unique given a further functionality axiom (Prop. 4.2), with propositional identity reducing to necessary equivalence (Prop. 3.2). The present statement differs in characterizing $\Box$ by objectivity rather than broadness alone and in assuming only predicative

Proof. By **T1**, **Imp**, **Ref**, and **LL** above, $\top$ satisfies every condition of **D8** except normality, which the *Rule of Equivalence* supplies. For necessitation, $\vdash \varphi$ makes φ and $\top$ interderivable, so $\vdash \varphi \equiv \top$, i.e., $\vdash \top\varphi$. For distribution, **LL** carries $(p \rightarrow q) \equiv \top$ to $(\top \rightarrow q) \equiv \top$ given $p \equiv \top$, which the rule's instance $(\top \rightarrow q) \equiv q$ collapses to $q \equiv \top$. Uniqueness up to equivalence follows from **L1** above. $\square$

Since the definitions and proofs above are schematic in $\mathcal{Q}$, they apply to any operator term; the characterization of metaphysical necessity is then the hypothesis $\text{Str}_L(\Box)$, under which **L1** and **T2** deliver uniqueness up to equivalence and the **S4** axiom for the primitive $\Box$.

5.2 Two-Dimensional Semantics

This section concerns the two-dimensional models introduced in §2.2. Definitions will be restated for convenience throughout.

D10 A *two-dimensional model* is a structure $\mathcal{M} = \langle W, T, \leq, |\cdot| \rangle$ where:

Worlds: A nonempty set of *worlds* W .

Times: A nonempty set of *times* T .

Order: A weak total order $\leq$ on T .

Interpretation: An interpretation function $|\cdot|$ assigning a set of world-time pairs $|p_i| \subseteq W \times T$ to each $p_i \in \mathbb{L}$.

D11 The language $\mathcal{L}^M := \langle \mathbb{L}, \perp, \rightarrow, \boxtimes, \boxplus, \boxminus \rangle$ where $\mathbb{L} := \{p_i : i \in \mathbb{N}\}$ is a countable set of sentence letters where the remaining symbols denote falsity, material implication, the disputed modal operator that Thomason reads ‘necessarily always’, the universal past tense operator, and the universal future tense operator, respectively.

D12 Truth in a two-dimensional model at a world and time is defined recursively:

(p_i) $\mathcal{M}, w, x \models p_i$ iff $\langle w, x \rangle \in |p_i|$.

$(\perp)$ $\mathcal{M}, w, x \not\models \perp$.

$(\rightarrow)$ $\mathcal{M}, w, x \models \varphi \rightarrow \psi$ iff $\mathcal{M}, w, x \not\models \varphi$ or $\mathcal{M}, w, x \models \psi$.

$(\boxtimes)$ $\mathcal{M}, w, x \models \boxtimes\varphi$ iff $\mathcal{M}, u, y \models \varphi$ for all $u \in W$ and $y \in T$.

$(\boxplus)$ $\mathcal{M}, w, x \models \boxplus\varphi$ iff $\mathcal{M}, w, y \models \varphi$ for all $y \in T$ where $y < x$.

$(\boxminus)$ $\mathcal{M}, w, x \models \boxminus\varphi$ iff $\mathcal{M}, w, y \models \varphi$ for all $y \in T$ where $x < y$.

comprehension over a minimal theory of $\equiv$. Given the Rule of Equivalence, propositional identity is coarse-grained enough to block Russell–Myhill on its own, and so the predicative restriction on comprehension could be dropped. However, neither the rule nor Booleanism is otherwise assumed.

D13 A nonempty relation $Z \subseteq (W_1 \times T_1) \times (W_2 \times T_2)$ — writing $(w, x) \rightarrow (v, y)$ to represent pairs in Z — is a $\mathcal{L}^M$ -bisimulation between $\mathcal{M}_1$ and $\mathcal{M}_2$ just in case whenever $(w, x) \rightarrow (w', x')$, all of the following conditions hold:

Atomic harmony: For all $p_i \in \mathbb{L}$, $\mathcal{M}_1, w, x \models p_i$ just in case $\mathcal{M}_2, w', x' \models p_i$.

Past Forth: For every y with $y < x$ there is y' with $y' < x'$ and $(w, y) \rightarrow (w', y')$.

Past Back: For every y' with $y' < x'$ there is y with $y < x$ and $(w, y) \rightarrow (w', y')$.

Future Forth: For every y with $x < y$ there is y' with $x' < y'$ and $(w, y) \rightarrow (w', y')$.

Future Back: For every y' with $x' < y'$ there is y with $x < y$ and $(w, y) \rightarrow (w', y')$.

Global Forth: For every $(u, z) \in W_1 \times T_1$ there is $(u', z') \in W_2 \times T_2$ with $(u, z) \rightarrow (u', z')$.

Global Back: For every $(u', z') \in W_2 \times T_2$ there is $(u, z) \in W_1 \times T_1$ with $(u, z) \rightarrow (u', z')$.

L3 If Z is a $\mathcal{L}^M$ -bisimulation between $\mathcal{M}_1$ and $\mathcal{M}_2$ and $(w, x) \rightarrow (w', x')$, then for every $\mathcal{L}^M$ -formula φ , $\mathcal{M}_1, w, x \models \varphi$ just in case $\mathcal{M}_2, w', x' \models \varphi$.

Proof. The proof proceeds by induction on the complexity of φ where the case for the sentence letters and extensional operators are routine.

Case $\Box$: Assume $\varphi = \Box\psi$. Supposing for contraposition that $\mathcal{M}_2, w', x' \not\models \Box\psi$, it follows that $\mathcal{M}_2, w', y' \not\models \psi$ for some $x' < y'$. By *Future Back*, there is a y with $x < y$ where $(w, y) \rightarrow (w', y')$, and so $\mathcal{M}_1, w, y \not\models \psi$ by hypothesis. Thus $\mathcal{M}_1, w, x \not\models \Box\psi$. Contraposition and parity of reasoning establish that $\mathcal{M}_1, w, x \models \Box\psi$ just in case $\mathcal{M}_2, w', x' \models \Box\psi$. The other tense operators are similar.

Case $\Box$: Assume $\varphi = \Box\psi$. Supposing for contraposition that $\mathcal{M}_2, w', x' \not\models \Box\psi$, it follows that $\mathcal{M}_2, u', z' \not\models \psi$ for some $u' \in W_2$ and $z' \in T_2$. By *Global Back*, there is $(u, z) \in W_1 \times T_1$ with $(u, z) \rightarrow (u', z')$, and so $\mathcal{M}_1, u, z \not\models \psi$ by hypothesis. Thus $\mathcal{M}_1, w, x \not\models \Box\psi$, where contraposition and parity of reasoning complete the case.

By induction on complexity, we may conclude that $\mathcal{M}_1, w, x \models \varphi$ just in case $\mathcal{M}_2, w', x' \models \varphi$ for all well-formed sentences φ in $\mathcal{L}^M$. $\square$

T3 $\Box$ is not definable in $\mathcal{L}^M$.

Proof. Assume for contradiction that $\Box$ is definable in $\mathcal{L}^M$ so that $\Box\varphi$ abbreviates a well-formed sentence of $\mathcal{L}^M$ with the following derived semantic clause:

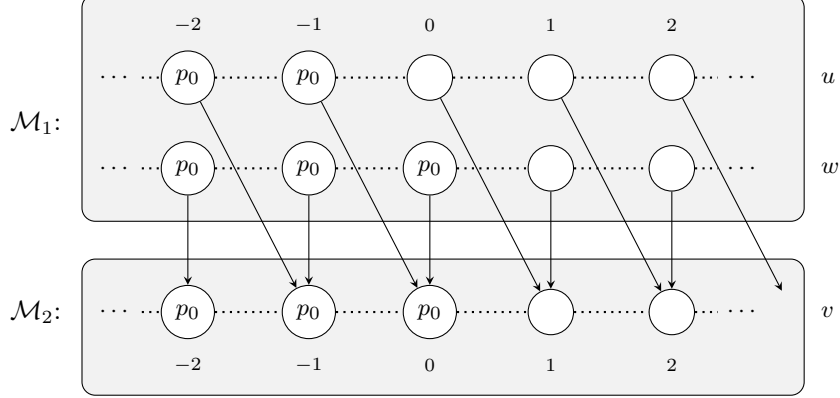
($\Box$) $\mathcal{M}, w, x \models \Box\varphi$ iff $\mathcal{M}, u, x \models \varphi$ for all $u \in W$.

We define the models $\mathcal{M}_1 = \langle W_1, T_1, \leq, |\cdot|_1 \rangle$ and $\mathcal{M}_2 = \langle W_2, T_2, \leq, |\cdot|_2 \rangle$ where:

$\mathcal{M}_1$: $W_1 = \{w, u\}$, $T_1 = \mathbb{Z}$, and $|p_0|_1 = \{(w, x) : x \leq 0\} \cup \{(u, x) : x \leq -1\}$, so that w satisfies p_0 at every time up to and including 0, while u satisfies p_0 at every time up to and including -1 .

$\mathcal{M}_2$: $W_2 = \{v\}$, $T_2 = \mathbb{Z}$, and $|p_0|_2 = \{(v, x) : x \leq 0\}$, so that v satisfies p_0 at every time up to and including 0.

We may then define the relation $Z \subseteq (W_1 \times T_1) \times (W_2 \times T_2)$, writing $(w, x) \rightarrow (v, y)$ to represent pairs in Z as before, by $(w, x) \rightarrow (v, x)$ for every $x \in \mathbb{Z}$ and $(u, x) \rightarrow (v, x + 1)$ for every $x \in \mathbb{Z}$. The following diagram depicts a representative window of the construction, where the ellipses indicate that the pattern continues without bound in both directions:



We verify that Z is a $\mathcal{L}^M$ -bisimulation by checking the clauses of [D13](#) for an arbitrary pair in Z as given below:

Atomic harmony: For $(w, x) \rightarrow (v, x)$: $\mathcal{M}_1, w, x \models p_0$ just in case $x \leq 0$, and $\mathcal{M}_2, v, x \models p_0$ just in case $x \leq 0$. For $(u, x) \rightarrow (v, x + 1)$: $\mathcal{M}_1, u, x \models p_0$ just in case $x \leq -1$, and $\mathcal{M}_2, v, x + 1 \models p_0$ just in case $x + 1 \leq 0$, i.e. $x \leq -1$. Both pairs agree on p_0 in every case.

Past Forth and Past Back: For $(w, x) \rightarrow (v, x)$, every $z < x$ satisfies $(w, z) \rightarrow (v, z)$ with $z < x$, giving both directions by taking $z' = z$. For $(u, x) \rightarrow (v, x + 1)$, every $z < x$ satisfies $(u, z) \rightarrow (v, z + 1)$ with $z + 1 < x + 1$, giving Past Forth; conversely every $z' < x + 1$ in T_2 satisfies $z' - 1 < x$ with $(u, z' - 1) \rightarrow (v, z')$, giving Past Back.

Future Forth and Future Back: Symmetric to the previous case: for $(w, x) \rightarrow (v, x)$, take $z' = z$ for every $z > x$ and conversely; for $(u, x) \rightarrow (v, x + 1)$, every $z > x$ satisfies $(u, z) \rightarrow (v, z + 1)$ with $z + 1 > x + 1$, and every $z' > x + 1$ satisfies $z' - 1 > x$ with $(u, z' - 1) \rightarrow (v, z')$.

Global Forth: Every $(w, x) \in W_1 \times T_1$ appears via $(w, x) \rightarrow (v, x)$, and every (u, x) appears via $(u, x) \rightarrow (v, x + 1)$.

Global Back: Every $(v, x) \in W_2 \times T_2$ is the right-projection of $(w, x) \rightarrow (v, x)$.

Thus Z is a $\mathcal{L}^M$ -bisimulation. Given that $(w, 0) \rightarrow (v, 0)$, it follows by **L3** that for all well-formed φ of $\mathcal{L}^M$, the following biconditional holds:

$$\mathcal{M}_1, w, 0 \models \varphi \text{ just in case } \mathcal{M}_2, v, 0 \models \varphi. \quad (*)$$

On the assumption that $\Box$ is definable in $\mathcal{L}^M$, we may conclude that $\mathcal{M}_1, w, 0 \models \Box p_0$ just in case $\mathcal{M}_2, v, 0 \models \Box p_0$. By the claimed semantic clause for $\Box$ we have:

$$\begin{aligned} \mathcal{M}_1, w, 0 \models \Box p_0 &\Leftrightarrow \mathcal{M}_1, w, 0 \models p_0 \text{ and } \mathcal{M}_1, u, 0 \models p_0 \\ \mathcal{M}_2, v, 0 \models \Box p_0 &\Leftrightarrow \mathcal{M}_2, v, 0 \models p_0 \end{aligned}$$

Since $\mathcal{M}_1, u, 0 \not\models p_0$, it follows that $\mathcal{M}_1, w, 0 \not\models \Box p_0$. Given that v is the only world in W_2 and $\mathcal{M}_2, v, 0 \models p_0$, it follows that $\mathcal{M}_2, v, 0 \models \Box p_0$. Hence $\mathcal{M}_1, w, 0 \not\models \Box p_0$ while $\mathcal{M}_2, v, 0 \models \Box p_0$, contradicting (*). Thus we may conclude by *reductio* that the operator $\Box$ with the derived semantic clause given above is not definable in $\mathcal{L}^M$. $\square$

D14 Dorra and Goodman [29] compensate for the limited expressive power of $\mathcal{L}^M$ by adding a countable set of time variables $\mathcal{V} := \{t_i : i \in \mathbb{N}\}$ which may be bound by first-order quantifiers, thereby obtaining the language $\mathcal{L}^F$. An *assignment* is a function $g : \mathcal{V} \rightarrow T$ from time variables in $\mathcal{V}$ to times in T which is used to extend the semantics. Truth in a two-dimensional model at a world and time relative to an assignment extends the semantics for $\mathcal{L}^M$ to include the following clauses:

- ($\exists t$) $\mathcal{M}, w, x, g \models \exists t \varphi$ iff $\mathcal{M}, w, x, g' \models \varphi$ for some g' differing from g at most in t .
- (Present) $\mathcal{M}, w, x, g \models \text{Present}(t)$ iff $g(t) = x$.

L4 In $\mathcal{L}^F$, $\Box$ may be defined by $\Box \varphi := \exists t [\text{Present}(t) \wedge \Box(\text{Present}(t) \rightarrow \varphi)]$.

Proof. To establish that $\Box$ is definable, we show:

$$\begin{aligned} \mathcal{M}, w, x, g \models \Box [\text{Present}(t) \wedge \Box(\text{Present}(t) \rightarrow \varphi)] \\ \Leftrightarrow \text{there is } g' \text{ with } g'(t) = x \text{ and } \mathcal{M}, u, y, g' \models \text{Present}(t) \rightarrow \varphi \text{ for all } u \in W, y \in T \\ \Leftrightarrow \text{there is } g' \text{ with } g'(t) = x \text{ and } \mathcal{M}, u, x, g' \models \varphi \text{ for all } u \in W \\ \Leftrightarrow \mathcal{M}, u, x, g \models \varphi \text{ for all } u \in W \end{aligned}$$

The first equivalence follows from the semantics for $\exists$, $\wedge$, and $\Box$. For the second equivalence, since $g'(t) = x$, we have $\mathcal{M}, u, y, g' \models \text{Present}(t) \rightarrow \varphi$ just in case either $y \neq x$ or $\mathcal{M}, u, y, g' \models \varphi$ with $y = x$. It follows that $\mathcal{M}, u, y, g' \models \text{Present}(t) \rightarrow \varphi$ for all $u \in W$ and $y \in T$ just in case $\mathcal{M}, u, x, g' \models \varphi$ for all $u \in W$. The final equivalence holds because t does not occur free in φ , so g and g' agree on the truth of φ , and so we may choose any t -variant g' of g with $g'(t) = x$ to witness the existential. $\square$

T4 Given the two-dimensional semantics for $\mathcal{L}^F$, both **P1** and **P2** are invalid.

Proof. Let $\mathcal{M} = \langle W, T, \leq, |\cdot| \rangle$ be a two-dimensional model with worlds $W = \{w, u\}$, times $T = \{0, 1\}$ where $\leq$ is the usual order on $\{0, 1\}$, and $|p_0| = \{(w, 0), (u, 0)\}$. Since $(w, 0) \in |p_0|$ and $(u, 0) \in |p_0|$, we have $\mathcal{M}, w, 0, g \models p_0$ and $\mathcal{M}, u, 0, g \models p_0$ where g is any assignment. It follows by **L4** that $\mathcal{M}, w, 0, g \models \Box p_0$.

However, $\mathcal{M}, w, 0, g \models \Box p_0$ just in case $\mathcal{M}, w, y, g \models p_0$ for all $y \in T$ with $0 \leq y$. Since $0 \leq 1$ and $(w, 1) \notin |p_0|$, we have $\mathcal{M}, w, 1, g \not\models p_0$. Therefore $\mathcal{M}, w, 0, g \not\models \Box p_0$. It follows that $\mathcal{M}, w, 0, g \not\models \Box p_0 \rightarrow \Box p_0$, and so **P1** is invalid. Since **P2** is equivalent to **P1**, it follows that **P2** is also invalid. $\square$

D15 An *order automorphism* on the structure $\langle T, \leq \rangle$ is a bijective function $\bar{a} : T \rightarrow T$ such that for all $x, y \in T$, we have $x \leq y$ just in case $\bar{a}(x) \leq \bar{a}(y)$.

D16 Given a two-dimensional model $\mathcal{M} = \langle W, T, \leq, |\cdot| \rangle$, the worlds $w, w' \in W$ are *time-shifted from x to y* —written $w \approx_x^y w'$ —iff there exists an order automorphism $\bar{a} : T \rightarrow T$ where $y = \bar{a}(x)$ and for all sentence letters $p_i \in \mathbb{L}$ and times $z \in T$, we have $\langle w, z \rangle \in |p_i|$ just in case $\langle w', \bar{a}(z) \rangle \in |p_i|$.

L5 If $w \approx_x^y u$ for some $x, y \in T$, then there is some order automorphism $\bar{a} : T \rightarrow T$ where $w \approx_z^{\bar{a}(z)} u$ for all $z \in T$.

Proof. Suppose $w \approx_x^y u$ for $x, y \in T$. By **D16**, $y = \bar{a}(x)$ for some order automorphism $\bar{a} : T \rightarrow T$ where the following equivalence holds for all $p_i \in \mathbb{L}$ and $z \in T$:

$$\langle w, z \rangle \in |p_i| \text{ just in case } \langle u, \bar{a}(z) \rangle \in |p_i|. \quad (*)$$

Let $z \in T$ be arbitrary. To show $w \approx_z^{\bar{a}(z)} u$, we must exhibit an order automorphism $\bar{b} : T \rightarrow T$ satisfying two conditions: (i) $\bar{a}(z) = \bar{b}(z)$, and (ii) for all $p_i \in \mathbb{L}$ and $t \in T$, $\langle w, t \rangle \in |p_i|$ just in case $\langle u, \bar{b}(t) \rangle \in |p_i|$. Taking $\bar{b} = \bar{a}$ satisfies both conditions: (i) holds trivially, and (ii) is exactly (*). Since z was arbitrary, $w \approx_z^{\bar{a}(z)} u$ for all $z \in T$. $\square$

D17 A two-dimensional model $\mathcal{M} = \langle W, T, \leq, |\cdot| \rangle$ of $\mathcal{L}^K$ is *abundant* just in case for every $w \in W$ and times $x, y \in T$, there is some world $w' \in W$ where $w \approx_x^y w'$.

L6 $\mathcal{M}_2, w, x \models \varphi$ just in case $\mathcal{M}_2, u, y \models \varphi$ for any well-formed sentence φ of $\mathcal{L}^K$ and abundant two-dimensional model $\mathcal{M}_2 = \langle W, T, \leq, |\cdot| \rangle$ where $w \approx_x^y u$.

Proof. Assume $w \approx_x^y u$ in $\mathcal{M}_2 = \langle W, T, \leq, |\cdot| \rangle$. The proof proceeds by induction on the complexity of φ . By **D16**, there exists an order automorphism $\bar{a} : T \rightarrow T$ where $y = \bar{a}(x)$ and for all sentence letters $p_i \in \mathbb{L}$ and times $z \in T$:

$$\langle w, z \rangle \in |p_i| \Leftrightarrow \langle u, \bar{a}(z) \rangle \in |p_i|. \quad (\dagger)$$

Base Case ($\varphi = p_i$): By $(\dagger)$ with $z = x$, it follows that $\langle w, x \rangle \in |p_i|$ just in case $\langle u, \bar{a}(x) \rangle \in |p_i|$. Since $y = \bar{a}(x)$, this gives $\langle w, x \rangle \in |p_i|$ just in case $\langle u, y \rangle \in |p_i|$. By the semantic clause for sentence letters, $\mathcal{M}_2, w, x \models p_i$ just in case $\mathcal{M}_2, u, y \models p_i$.

Base Case ($\varphi = \perp$): By the semantic clause for $\perp$, we have $\mathcal{M}_2, w, x \not\models \perp$ and $\mathcal{M}_2, u, y \not\models \perp$. Therefore the biconditional holds immediately.

Inductive Case ($\varphi = \psi \rightarrow \chi$):

$$\begin{aligned}\mathcal{M}_2, w, x \models \psi \rightarrow \chi &\Leftrightarrow \mathcal{M}_2, w, x \not\models \psi \text{ or } \mathcal{M}_2, w, x \models \chi \\ &\Leftrightarrow \mathcal{M}_2, u, y \not\models \psi \text{ or } \mathcal{M}_2, u, y \models \chi \\ &\Leftrightarrow \mathcal{M}_2, u, y \models \psi \rightarrow \chi\end{aligned}$$

The induction hypothesis justifies the second equivalence when applied to ψ and χ .

Inductive Case ($\varphi = \Box\psi$):

$$\begin{aligned}\mathcal{M}_2, w, x \not\models \Box\psi &\Leftrightarrow \mathcal{M}_2, v, x \not\models \psi \text{ for some } v \in W \\ &\Leftrightarrow \mathcal{M}_2, v', y \not\models \psi \text{ for some } v \in W \\ &\Leftrightarrow \mathcal{M}_2, u, y \not\models \Box\psi\end{aligned}$$

Since $\mathcal{M}_2$ is abundant where $v \in W$ and $x, y \in T$, there is some $v' \in W$ where $v \approx_x^y v'$. It follows by the induction hypothesis that $\mathcal{M}_2, v', y \not\models \psi$, where parity of reasoning establishes the converse. This justifies the second biconditional.

Inductive Case ($\varphi = \Box\psi$):

$$\begin{aligned}\mathcal{M}_2, w, x \not\models \Box\psi &\Leftrightarrow \mathcal{M}_2, w, z \not\models \psi \text{ for some } z < x \\ &\Leftrightarrow \mathcal{M}_2, u, z' \not\models \psi \text{ for some } z' < y \\ &\Leftrightarrow \mathcal{M}_2, u, y \not\models \Box\psi\end{aligned}$$

For the forward direction of the second biconditional, assume that $\mathcal{M}_2, w, z \not\models \psi$ for some $z < x$. Since $\bar{a}$ is an order automorphism, $\bar{a}(z) < \bar{a}(x)$ where $\bar{a}(x) = y$. Letting $z' = \bar{a}(z)$, we have $z' < y$. Since $w \approx_x^y u$ where $y = \bar{a}(x)$ and $z' = \bar{a}(z)$, it follows by **L5** that $w \approx_z^{z'} u$. Applying the induction hypothesis to ψ , we have $\mathcal{M}_2, w, z \models \psi$ just in case $\mathcal{M}_2, u, z' \models \psi$, and so $\mathcal{M}_2, u, z' \not\models \psi$ for some $z' < y$.

For the backward direction of the second biconditional, assume $\mathcal{M}_2, u, z' \not\models \psi$ for some $z' < y$. Since $\bar{a} : T \rightarrow T$ is an order automorphism, we have $\bar{a}^{-1}(z') < \bar{a}^{-1}(y)$. Letting $z = \bar{a}^{-1}(z')$, we have $z < x$ given that $x = \bar{a}^{-1}(y)$. Since $w \approx_x^y u$ where $y = \bar{a}(x)$ and $z' = \bar{a}(z)$, we get $w \approx_z^{z'} u$ by **L5**, and so $\mathcal{M}_2, w, z \models \psi$ just in case $\mathcal{M}_2, u, z' \models \psi$ by the induction hypothesis. Thus we may conclude that $\mathcal{M}_2, w, z \not\models \psi$ for some $z < x$, completing the case.

Inductive Case ($\varphi = \Box\psi$): The proof is analogous to the case for $\Box\psi$, using $x < z$ and $y < z'$ in place of $z < x$ and $z' < y$. $\square$

T5 Both **P1** and **P2** are valid over a two-dimensional frame $\langle W, T, \leq \rangle$ with $W \neq \emptyset$ if and only if $|T| \leq 1$.

Proof. ($\Leftarrow$) Suppose $|T| \leq 1$. Since every two-dimensional model requires T to be nonempty (**D10**), we have $|T| = 1$; let $T = \{x\}$. Then $\Box\varphi$ requires $\mathcal{M}, w, y \models \varphi$ for all y with $x < y$, and $\Box\varphi$ requires $\mathcal{M}, w, y \models \varphi$ for all y with $y < x$; since $T = \{x\}$

contains no such y in either case, both $\Box\varphi$ and $\Box\varphi$ are vacuously true at $\mathcal{M}, w, x$ for every φ . Since $\Delta\varphi := \Box\varphi \wedge \varphi \wedge \Box\varphi$, it follows that $\mathcal{M}, w, x \models \Delta\varphi$ just in case $\mathcal{M}, w, x \models \varphi$. Given that $\Box\varphi$ entails $\mathcal{M}, u, x \models \varphi$ for all $u \in W$, we have in particular $\mathcal{M}, w, x \models \varphi$, and so $\mathcal{M}, w, x \models \Delta\varphi$. Therefore $\Box\varphi \rightarrow \Delta\varphi$ is valid, and **P2** follows by contraposition as above.

($\Rightarrow$) Suppose $|T| \geq 2$. Since $\leq$ is a weak total order on T , there exist distinct $x, y \in T$ with $x < y$. Let $w \in W$ be arbitrary, and define $|p_0| = W \times \{x\}$. For any $u \in W$, we have $(u, x) \in |p_0|$, and so $\mathcal{M}, u, x \models p_0$. By the semantic clause for $\Box$, it follows that $\mathcal{M}, w, x \models \Box p_0$. However, since $x < y$ and $(w, y) \notin |p_0|$, we have $\mathcal{M}, w, y \not\models p_0$. Since $x < y$, it follows that $\mathcal{M}, w, x \not\models \Box p_0$, and so $\mathcal{M}, w, x \not\models \Delta p_0$. Therefore $\mathcal{M}, w, x \not\models \Box p_0 \rightarrow \Delta p_0$, and **P1** is falsified. Since **P2** is equivalent to **P1**, it follows that **P2** is also falsified. $\square$

T6 Both **P1** and **P2** are valid over the abundant two-dimensional models of $\mathcal{L}^K$.

Proof. Let $\mathcal{M} = \langle W, T, \leq, |\cdot| \rangle$ be an abundant two-dimensional model of $\mathcal{L}^K$. Assume for *reductio* that $\mathcal{M}, w, x \not\models \Box\varphi \rightarrow \Delta\varphi$ for some $w \in W$, $x \in T$, and well-formed sentence φ of $\mathcal{L}^K$. Thus $\mathcal{M}, w, x \models \Box\varphi$ and $\mathcal{M}, w, x \not\models \Delta\varphi$.

By the semantic clause for $\Box$, we have $\mathcal{M}, u, x \models \varphi$ for all $u \in W$. In particular, $\mathcal{M}, w, x \models \varphi$. Given that $\Delta\varphi := \Box\varphi \wedge \varphi \wedge \Box\varphi$ and $\mathcal{M}, w, x \models \varphi$, it follows from $\mathcal{M}, w, x \not\models \Delta\varphi$ that either $\mathcal{M}, w, x \not\models \Box\varphi$ or $\mathcal{M}, w, x \not\models \Box\varphi$.

Case 1: Assume $\mathcal{M}, w, x \not\models \Box\varphi$. By the semantic clause for $\Box$, there exists $y \in T$ with $y < x$ such that $\mathcal{M}, w, y \not\models \varphi$. Since $\mathcal{M}$ is abundant, there exists $w' \in W$ such that $w \approx_y^x w'$. By **L6**, $\mathcal{M}, w, y \models \varphi$ just in case $\mathcal{M}, w', x \models \varphi$. Since $\mathcal{M}, w, y \not\models \varphi$, we have $\mathcal{M}, w', x \not\models \varphi$. This contradicts that $\mathcal{M}, u, x \models \varphi$ for all $u \in W$.

Case 2: Assume $\mathcal{M}, w, x \not\models \Box\varphi$. By the semantic clause for $\Box$, there exists $z \in T$ with $x < z$ such that $\mathcal{M}, w, z \not\models \varphi$. Since $\mathcal{M}$ is abundant, there exists $w'' \in W$ such that $w \approx_z^x w''$. By **L6**, $\mathcal{M}, w, z \models \varphi$ just in case $\mathcal{M}, w'', x \models \varphi$. Since $\mathcal{M}, w, z \not\models \varphi$, we have $\mathcal{M}, w'', x \not\models \varphi$. This contradicts that $\mathcal{M}, u, x \models \varphi$ for all $u \in W$.

Both cases yield a contradiction. Therefore $\mathcal{M}, w, x \models \Box\varphi \rightarrow \Delta\varphi$ for all abundant models $\mathcal{M}$, worlds $w \in W$, times $x \in T$, and well-formed sentences φ of $\mathcal{L}^K$. Hence **P1** is valid over the abundant two-dimensional models of $\mathcal{L}^K$.

Since $\Box\neg\varphi \rightarrow \Delta\neg\varphi$ by **P1**, contraposition yields $\neg\Delta\neg\varphi \rightarrow \neg\Box\neg\varphi$. By definition, $\nabla\varphi \rightarrow \Diamond\varphi$, and so **P2** is valid over the abundant two-dimensional models of $\mathcal{L}^K$. $\square$

T7 Abundant models with at least two distinct times are unbounded.

Proof. Let $\mathcal{M} = \langle W, T, \leq, |\cdot| \rangle$ be an abundant two-dimensional model of $\mathcal{L}^K$ where $x < y$ for $x, y \in T$. We prove that T is unbounded below and unbounded above.

Below: Assume for *reductio* that T is bounded below. Thus there is some $t_{\min} \in T$ where $t_{\min} \leq z$ for all $z \in T$. Since $t_{\min} \leq x < y$, either $t_{\min} < x$ or $t_{\min} = x < y$. In either case, there exists $t \in \{x, y\}$ with $t_{\min} < t$. Letting $w \in W$, it follows by abundance that there is some $w' \in W$ such that $w \approx_{t_{\min}}^t w'$. By definition, there is an order automorphism $\bar{a} : T \rightarrow T$ where $t = \bar{a}(t_{\min})$. Since $t_{\min} \leq z$ for all $z \in T$ and $\bar{a}$ is order-preserving, we have $\bar{a}(t_{\min}) \leq \bar{a}(z)$ for all $z \in T$. Since $\bar{a}$ is surjective, for any

$z' \in T$ there exists $z \in T$ where $\bar{a}(z) = z'$. Thus $\bar{a}(t_{\min}) \leq z'$ for all $z' \in T$, so $\bar{a}(t_{\min})$ is minimal in T . Therefore $\bar{a}(t_{\min}) = t_{\min}$, and so $t = t_{\min}$, which contradicts $t_{\min} < t$. We may conclude by *reductio* that T is unbounded below.

Above: Assume for *reductio* that T is bounded above. Thus there is some $t_{\max} \in T$ where $z \leq t_{\max}$ for all $z \in T$. Since $x < y \leq t_{\max}$, either $y < t_{\max}$ or $x < y = t_{\max}$. In either case, there exists $t \in \{x, y\}$ with $t < t_{\max}$. Letting $w \in W$, it follows by abundance that there is some $w' \in W$ such that $w \approx_{t_{\max}}^t w'$. By definition, there is an order automorphism $\bar{a} : T \rightarrow T$ where $t = \bar{a}(t_{\max})$. Since $z \leq t_{\max}$ for all $z \in T$ and $\bar{a}$ is order-preserving, we have $\bar{a}(z) \leq \bar{a}(t_{\max})$ for all $z \in T$. Since $\bar{a}$ is surjective, for any $z' \in T$ there exists $z \in T$ where $\bar{a}(z) = z'$. Thus $z' \leq \bar{a}(t_{\max})$ for all $z' \in T$, so $\bar{a}(t_{\max})$ is maximal in T . Therefore $\bar{a}(t_{\max}) = t_{\max}$, and so $t = t_{\max}$, which contradicts $t < t_{\max}$. We may conclude by *reductio* that T is unbounded above. $\square$

5.3 Task Semantics

We begin this section by restating a number of definitions for convenience:

D18 The language $\mathcal{L} := \langle \mathbb{L}, \perp, \rightarrow, \Box, \Box, \Box \rangle$ where $\mathbb{L} := \{p_i : i \in \mathbb{N}\}$ is a countable set of sentence letters and the remaining symbols denote falsity, material implication, the metaphysical necessity operator, the universal past tense operator, and the universal future tense operator, respectively.

D19 A *frame* is a structure $\mathcal{F} = \langle W, \mathcal{D}, \Rightarrow \rangle$ where:

World States: A nonempty set of *world states* W .

Temporal Order: A totally ordered abelian group $\mathcal{D} = \langle D, +, 0, \leq \rangle$.

Task Relation: A parameterized task relation $\Rightarrow$ satisfying:

Nullity: $w \Rightarrow_0 w$.

Reflection: If $w \Rightarrow_x u$, then $u \Rightarrow_{-x} w$.

Compositionality: If $w \Rightarrow_x u$ and $u \Rightarrow_y v$, then $w \Rightarrow_{x+y} v$.

D20 A *world history* over a frame $\mathcal{F} = \langle W, \mathcal{D}, \Rightarrow \rangle$ is a function $\tau : X \rightarrow W$ where $X \subseteq D$ is a nonempty convex set and $\tau(x) \Rightarrow_{y-x} \tau(y)$ for all times $x, y \in X$. The set of all world histories over $\mathcal{F}$ is denoted $H_{\mathcal{F}}$ which I will henceforth refer to as *possible worlds*.

D21 A *model* of $\mathcal{L}$ is a structure $\mathcal{M} = \langle W, \mathcal{D}, \Rightarrow, |\cdot| \rangle$ where $\mathcal{F} = \langle W, \mathcal{D}, \Rightarrow \rangle$ is a frame and $|p_i| \subseteq W$ for every sentence letter $p_i \in \mathbb{L}$.

D22 Truth in a model at a possible world and time is defined recursively:

(p_i) $\mathcal{M}, \tau, x \models p_i$ iff $x \in \text{dom}(\tau)$ and $\tau(x) \in |p_i|$.

$(\perp)$ $\mathcal{M}, \tau, x \not\models \perp$.

- ($\rightarrow$) $\mathcal{M}, \tau, x \models \varphi \rightarrow \psi$ iff $\mathcal{M}, \tau, x \not\models \varphi$ or $\mathcal{M}, \tau, x \models \psi$.
- ($\Box$) $\mathcal{M}, \tau, x \models \Box\varphi$ iff $\mathcal{M}, \sigma, x \models \varphi$ for all $\sigma \in H_{\mathcal{F}}$.
- ($\Box_P$) $\mathcal{M}, \tau, x \models \Box_P\varphi$ iff $\mathcal{M}, \tau, y \models \varphi$ for all $y \in D$ where $y < x$.
- ($\Box_F$) $\mathcal{M}, \tau, x \models \Box_F\varphi$ iff $\mathcal{M}, \tau, y \models \varphi$ for all $y \in D$ where $x < y$.

D23 Given a frame $\mathcal{F} = \langle W, \mathcal{D}, \Rightarrow \rangle$, possible worlds $\tau, \sigma \in H_{\mathcal{F}}$ are *time-shifted from x to y* —written $\tau \approx_x^y \sigma$ —iff there exists an order automorphism $\bar{a} : D \rightarrow D$ where $y = \bar{a}(x)$, $\text{dom}(\sigma) = \bar{a}(\text{dom}(\tau))$, and $\tau(z) = \sigma(\bar{a}(z))$ for all $z \in \text{dom}(\tau)$.

L7 For any frame $\mathcal{F} = \langle W, \mathcal{D}, \Rightarrow \rangle$, possible world $\tau \in H_{\mathcal{F}}$, and times $x, y \in D$, there is a possible world $\sigma \in H_{\mathcal{F}}$ where $\tau \approx_x^y \sigma$ is witnessed by $\bar{a}(z) = z - x + y$.

Proof. Let $\mathcal{F} = \langle W, \mathcal{D}, \Rightarrow \rangle$ be a frame, $\tau \in H_{\mathcal{F}}$ a possible world where $x, y \in D$ are arbitrary times. Define $\bar{a} : D \rightarrow D$ by $\bar{a}(z) = z - x + y$.

Since $\mathcal{D} = \langle D, +, 0, \leq \rangle$ is an abelian group, $\bar{a}$ is a bijection. Supposing $z_1 \leq z_2$, it follows that $z_1 - x + y \leq z_2 - x + y$, and so $\bar{a}(z_1) \leq \bar{a}(z_2)$. Thus $\bar{a}$ is an order automorphism with $y = \bar{a}(x)$ where the inverse is $\bar{a}^{-1}(z) = z + x - y$.

Let $Y = \bar{a}(X)$ and define $\sigma(z) = \tau(\bar{a}^{-1}(z))$ for all $z \in D$ where $\bar{a}^{-1}(z) \in X$. Suppose $z_1, z_2 \in Y$ with $z_1 \leq z \leq z_2$, so $\bar{a}^{-1}(z_1), \bar{a}^{-1}(z_2) \in X$ and $\bar{a}^{-1}(z_1) \leq \bar{a}^{-1}(z) \leq \bar{a}^{-1}(z_2)$. Since $\bar{a}^{-1}(z) \in X$ by convexity, it follows that $z \in Y$, and so Y is convex.

Suppose $z_1, z_2 \in Y$. Then $\bar{a}^{-1}(z_1), \bar{a}^{-1}(z_2) \in X$ where:

$$\begin{aligned} \bar{a}^{-1}(z_2) - \bar{a}^{-1}(z_1) &= (z_2 + x - y) - (z_1 + x - y) \\ &= z_2 - z_1. \end{aligned}$$

Since τ is a possible world, $\tau(\bar{a}^{-1}(z_1)) \Rightarrow_{z_2 - z_1} \tau(\bar{a}^{-1}(z_2))$, and so $\sigma(z_1) \Rightarrow_{z_2 - z_1} \sigma(z_2)$. Thus $\sigma \in H_{\mathcal{F}}$ and $\tau \approx_x^y \sigma$ is witnessed by $\bar{a}$. $\square$

L8 $\mathcal{M}, \tau, x \models \varphi$ just in case $\mathcal{M}, \sigma, y \models \varphi$ for any model $\mathcal{M} = \langle W, \mathcal{D}, \Rightarrow, |\cdot| \rangle$ of $\mathcal{L}$, well-formed sentence φ of $\mathcal{L}$, and possible worlds $\tau, \sigma \in H_{\mathcal{F}}$ where $\tau \approx_x^y \sigma$ is witnessed by the time-shift function $\bar{a}(z) = z - x + y$.

Proof. We proceed by induction on the complexity of φ .

Base Case (p_i): The first and third biconditionals follow from **D22**.

$$\begin{aligned} \mathcal{M}, \tau, x \models p_i &\Leftrightarrow x \in \text{dom}(\tau) \text{ and } \tau(x) \in |p_i| \\ &\Leftrightarrow y \in \text{dom}(\sigma) \text{ and } \sigma(y) \in |p_i| \\ &\Leftrightarrow \mathcal{M}, \sigma, y \models p_i \end{aligned}$$

For the second biconditional, $\text{dom}(\sigma) = \bar{a}(\text{dom}(\tau))$ by **D23**, so $y = \bar{a}(x) \in \text{dom}(\sigma)$ just in case $x \in \text{dom}(\tau)$. When $x \in \text{dom}(\tau)$ and $y \in \text{dom}(\sigma)$, we have $\tau(x) = \sigma(\bar{a}(x)) = \sigma(y)$, and so the second biconditional holds. Otherwise, the second biconditional holds since $x \in \text{dom}(\tau)$ just in case $y \in \text{dom}(\sigma)$ given that $\text{dom}(\sigma) = \bar{a}(\text{dom}(\tau))$ where $y = \bar{a}(x)$.

Base Case ($\perp$): By **D22**, $\mathcal{M}, \tau, x \not\models \perp$ and $\mathcal{M}, \sigma, y \not\models \perp$.

Inductive Case ($\varphi \rightarrow \psi$): The first and third biconditionals follow from **D22** and the second biconditional follows from the inductive hypothesis.

$$\begin{aligned}\mathcal{M}, \tau, x \models \varphi \rightarrow \psi &\Leftrightarrow \mathcal{M}, \tau, x \not\models \varphi \text{ or } \mathcal{M}, \tau, x \models \psi \\ &\Leftrightarrow \mathcal{M}, \sigma, y \not\models \varphi \text{ or } \mathcal{M}, \sigma, y \models \psi \\ &\Leftrightarrow \mathcal{M}, \sigma, y \models \varphi \rightarrow \psi\end{aligned}$$

Inductive Case ($\Box\varphi$): The first and third biconditionals follow from **D22**.

$$\begin{aligned}\mathcal{M}, \tau, x \not\models \Box\varphi &\Leftrightarrow \mathcal{M}, \rho, x \not\models \varphi \text{ for some } \rho \in H_{\mathcal{F}} \\ &\Leftrightarrow \mathcal{M}, \rho', y \not\models \varphi \text{ for some } \rho' \in H_{\mathcal{F}} \\ &\Leftrightarrow \mathcal{M}, \sigma, y \not\models \Box\varphi\end{aligned}$$

To justify the forward direction of the second biconditional, assume $\rho \in H_{\mathcal{F}}$ with $\mathcal{M}, \rho, x \not\models \varphi$. By **L7** there is $\rho' \in H_{\mathcal{F}}$ where $\rho \approx_x^y \rho'$ is witnessed by $\bar{a}(z) = z - x + y$, so $\mathcal{M}, \rho', y \not\models \varphi$ by the inductive hypothesis. Now assume $\rho' \in H_{\mathcal{F}}$ with $\mathcal{M}, \rho', y \not\models \varphi$. By **L7**, there is $\rho \in H_{\mathcal{F}}$ where $\rho' \approx_y^x \rho$ is witnessed by $\bar{b}(z) = z - y + x$, so by the inductive hypothesis $\mathcal{M}, \rho, x \not\models \varphi$.

Inductive Case ($\Box\varphi$): The first and third biconditionals follow from **D22**.

$$\begin{aligned}\mathcal{M}, \tau, x \not\models \Box\varphi &\Leftrightarrow \mathcal{M}, \tau, x' \not\models \varphi \text{ for some } x' \in D \text{ where } x' < x \\ &\Leftrightarrow \mathcal{M}, \sigma, y' \not\models \varphi \text{ for some } y' \in D \text{ where } y' < y \\ &\Leftrightarrow \mathcal{M}, \sigma, y \not\models \Box\varphi\end{aligned}$$

To justify the forward direction of the second biconditional, assume $\mathcal{M}, \tau, x' \not\models \varphi$ for some $x' < x$. Since $\bar{a}$ is an order automorphism, we have $\bar{a}(x') < \bar{a}(x)$. Letting $y' = \bar{a}(x')$, it follows that $y' < y$. By assumption $\tau \approx_x^y \sigma$, and so $\text{dom}(\sigma) = \bar{a}(\text{dom}(\tau))$ and $\tau(z) = \sigma(\bar{a}(z))$ for all $z \in \text{dom}(\tau)$ by **D23**. Given that $\bar{a}(x') = y'$, the same $\bar{a}$ that witnesses $\tau \approx_x^y \sigma$ also witnesses $\tau \approx_{x'}^{y'} \sigma$. Thus $\mathcal{M}, \sigma, y' \not\models \varphi$ by hypothesis.

Now assume $\mathcal{M}, \sigma, y' \not\models \varphi$ for some $y' < y$. Since $\bar{a}$ is a bijection, there exists x' with $\bar{a}(x') = y'$. Since $\bar{a}$ is an order automorphism and $\bar{a}(x') = y' < y = \bar{a}(x)$, it follows that $x' < x$. It follows by the same reasoning above that $\bar{a}$ witnesses $\tau \approx_{x'}^{y'} \sigma$, and so by the inductive hypothesis $\mathcal{M}, \tau, x' \not\models \varphi$.

Inductive Case ($\Box\varphi$): The proof is similar to the $\Box$ case. $\square$

D24 A task frame $\mathcal{F} = \langle W, \mathcal{D}, \Rightarrow \rangle$ is:

DISCRETE if for any $x \in D$, whenever there exists $y > x$, there is a least such $y' > x$ satisfying $z \geq y'$ for all $z > x$.

DENSE if for any $x, y \in D$ where $x < y$, there exists $z \in D$ where $x < z < y$.

COMPLETE if every nonempty $S \subseteq D$ bounded above has a least upper bound in D .

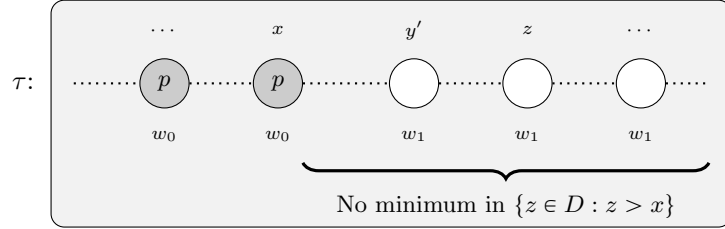
DETERMINISTIC if $u = v$ whenever $w \Rightarrow_x u$ and $w \Rightarrow_x v$ for $w, u, v \in W$ and $x \in D$.

D25 A well-formed sentence φ of $\mathcal{L}$ is *valid over a frame* $\mathcal{F} = \langle W, \mathcal{D}, \Rightarrow \rangle$ which we may write $\models_{\mathcal{F}} \varphi$ if and only if $\mathcal{M}, \tau, x \models \varphi$ for every model $\mathcal{M} = \langle W, \mathcal{D}, \Rightarrow, |\cdot| \rangle$ where $\mathcal{F} = \langle W, \mathcal{D}, \Rightarrow \rangle$, possible world $\tau \in H_{\mathcal{F}}$, and time $x \in D$.

T8 $(\Box\varphi \wedge \varphi \wedge \Diamond\top) \rightarrow \Diamond\Box\varphi$ is valid over every frame $\mathcal{F} = \langle W, \mathcal{D}, \Rightarrow \rangle$ if and only if $\mathcal{D}$ is DISCRETE.

Proof. ($\Leftarrow$) Suppose $\mathcal{D}$ is DISCRETE and let $\mathcal{F} = \langle W, \mathcal{D}, \Rightarrow \rangle$ be an arbitrary frame. Let $\mathcal{M} = \langle W, \mathcal{D}, \Rightarrow, |\cdot| \rangle$ be a model with $\tau \in H_{\mathcal{F}}$ and $x \in D$ where $\mathcal{M}, \tau, x \models \Box\varphi \wedge \varphi \wedge \Diamond\top$. From $\Diamond\top$, there exists some $y > x$. By **D24**, there is a least $x^+ > x$ where $z \geq x^+$ for all $z > x$. Let $t < x^+$ be arbitrary. Since x^+ is the least time greater than x , we know $t \leq x$. If $t < x$, then $\mathcal{M}, \tau, t \models \varphi$ by $\Box\varphi$. If $t = x$, then $\mathcal{M}, \tau, t \models \varphi$ directly. Thus $\mathcal{M}, \tau, x^+ \models \Box\varphi$, giving $\mathcal{M}, \tau, x \models \Diamond\Box\varphi$. Since $\mathcal{F}$, $\mathcal{M}$, τ , and x were arbitrary, $(\Box\varphi \wedge \varphi \wedge \Diamond\top) \rightarrow \Diamond\Box\varphi$ is valid over every frame $\mathcal{F} = \langle W, \mathcal{D}, \Rightarrow \rangle$.

($\Rightarrow$) Suppose $\mathcal{D}$ is not DISCRETE. Then there exists $x \in D$ such that $\{z \in D : z > x\}$ is nonempty but has no minimum. Let $W = \{w_0, w_1\}$ with $w \Rightarrow_d w'$ for all $w, w' \in W$ and $d \in D$, let $\mathcal{F}' = \langle W, \mathcal{D}, \Rightarrow \rangle$, and define $\mathcal{M} = \langle W, \mathcal{D}, \Rightarrow, |\cdot| \rangle$ with $|p| = \{w_0\}$ and $\tau : D \rightarrow W$ by $\tau(y) = w_0$ for $y \leq x$ and $\tau(y) = w_1$ for $y > x$. Since $\Rightarrow$ is universal, $\tau \in H_{\mathcal{F}'}$.



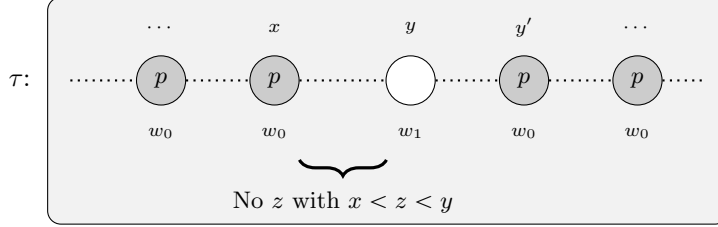
Thus $\mathcal{M}, \tau, x \models p$ since $\tau(x) = w_0 \in |p|$, and $\mathcal{M}, \tau, x \models \Box p$ since $\tau(y) = w_0 \in |p|$ for all $y < x$ by **D22**. Since $\{z \in D : z > x\}$ is nonempty, $\mathcal{M}, \tau, x \models \Diamond\top$. Suppose for contradiction that $\mathcal{M}, \tau, x \models \Diamond\Box p$. Then there exists $z > x$ with $\mathcal{M}, \tau, z \models \Box p$. Since there is no minimum in $\{z' \in D : z' > x\}$, there exists $y' \in D$ with $x < y' < z$. But $\tau(y') = w_1 \notin |p|$, so $\mathcal{M}, \tau, y' \not\models p$ by **D22**, contradicting $\mathcal{M}, \tau, z \models \Box p$. Therefore $\mathcal{M}, \tau, x \not\models \Diamond\Box p$, and so $\not\models_{\mathcal{F}'} (\Box\varphi \wedge \varphi \wedge \Diamond\top) \rightarrow \Diamond\Box\varphi$, and the axiom is not valid over every frame with temporal order $\mathcal{D}$. $\square$

T9 $\Box\Box\varphi \rightarrow \Box\varphi$ is valid over every frame $\mathcal{F} = \langle W, \mathcal{D}, \Rightarrow \rangle$ if and only if $\mathcal{D}$ is DENSE.

Proof. ($\Leftarrow$) Suppose $\mathcal{D}$ is DENSE and let $\mathcal{F} = \langle W, \mathcal{D}, \Rightarrow \rangle$ be an arbitrary frame. Let $\mathcal{M} = \langle W, \mathcal{D}, \Rightarrow, |\cdot| \rangle$ be a model with $\tau \in H_{\mathcal{F}}$ and $x \in D$ where $\mathcal{M}, \tau, x \models \Box\Box\varphi$. Assume for contradiction that $\mathcal{M}, \tau, x \not\models \Box\varphi$. Then there is some $y > x$ with $\mathcal{M}, \tau, y \not\models \varphi$ by **D22**. Since $\mathcal{D}$ is DENSE and $x < y$, there is a $z \in D$ where $x < z < y$ by **D24**. Since $\mathcal{M}, \tau, x \models \Box\Box\varphi$ and $z > x$, we have $\mathcal{M}, \tau, z \models \Box\varphi$ by **D22**. Thus we have

$\mathcal{M}, \tau, y \models \varphi$ since $y > z$, contradicting $\mathcal{M}, \tau, y \not\models \varphi$. Since $\mathcal{F}$, $\mathcal{M}$, τ , and x were arbitrary, $\Box\Box\varphi \rightarrow \Box\varphi$ is valid over every frame $\mathcal{F} = \langle W, \mathcal{D}, \Rightarrow \rangle$.

($\Rightarrow$) Suppose $\mathcal{D}$ is not DENSE. Then there exist $x, y \in D$ with $x < y$ and no $z \in D$ satisfying $x < z < y$. Let $W = \{w_0, w_1\}$ with $w \Rightarrow_d w'$ for all $w, w' \in W$ and $d \in D$, let $\mathcal{F}' = \langle W, \mathcal{D}, \Rightarrow \rangle$, and define $\mathcal{M} = \langle W, \mathcal{D}, \Rightarrow, |\cdot| \rangle$ with $|p| = \{w_0\}$ and $\tau : D \rightarrow W$ by $\tau(y) = w_1$ and $\tau(z) = w_0$ for all $z \neq y$. Since $\Rightarrow$ is universal, $\tau \in H_{\mathcal{F}'}$.



Given that $y > x$ and $\tau(y) = w_1 \notin |p|$, we have $\mathcal{M}, \tau, x \not\models \Box p$ by **D22**. Let $y' > x$ be arbitrary. Since there is no z with $x < z < y$, it follows that $y' \geq y$. Let $z' > y'$ be arbitrary. Then $z' > y$, so $z' \neq y$ and $\tau(z') = w_0 \in |p|$, giving $\mathcal{M}, \tau, z' \models p$. Since $z' > y'$ was arbitrary, $\mathcal{M}, \tau, y' \models \Box p$. Since $y' > x$ was arbitrary, $\mathcal{M}, \tau, x \models \Box\Box p$ by **D22**. Therefore $\mathcal{M}, \tau, x \not\models \Box\Box p \rightarrow \Box p$, and so $\not\models_{\mathcal{F}'} \Box\Box\varphi \rightarrow \Box\varphi$, and the axiom is not valid over every frame with temporal order $\mathcal{D}$. $\square$

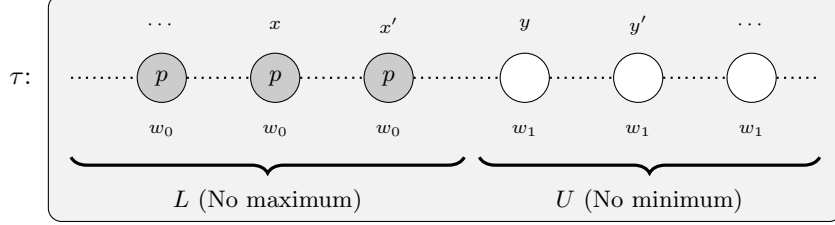
T10 $\Delta(\Box\varphi \rightarrow \Diamond\Box\varphi) \rightarrow (\Box\varphi \rightarrow \Box\varphi)$ is valid over every frame $\mathcal{F} = \langle W, \mathcal{D}, \Rightarrow \rangle$ if and only if $\mathcal{D}$ is COMPLETE.

Proof. ($\Leftarrow$) Suppose $\mathcal{D}$ is COMPLETE and let $\mathcal{F} = \langle W, \mathcal{D}, \Rightarrow \rangle$ be an arbitrary frame. Let $\mathcal{M} = \langle W, \mathcal{D}, \Rightarrow, |\cdot| \rangle$ be a model with $\mathcal{M}, \tau, x \models \Delta(\Box\varphi \rightarrow \Diamond\Box\varphi)$ and $\mathcal{M}, \tau, x \models \Box\varphi$ for $\tau \in H_{\mathcal{F}}$ and $x \in D$. Suppose for contradiction that $\mathcal{M}, \tau, x \not\models \Box\varphi$. Then there exists some $y > x$ with $\mathcal{M}, \tau, y \not\models \varphi$. Let $B = \{z \in D : z > x \text{ and } \mathcal{M}, \tau, z \not\models \varphi\}$, which is nonempty. Since B is bounded below by x , the set $\{-z : z \in B\}$ is bounded above by $-x$, so by **D24** it has a least upper bound, and $c = \inf(B)$ exists in D with $c \geq x$. Since $\mathcal{M}, \tau, x \models \Box\varphi$ and $\mathcal{M}, \tau, x \models \Box\varphi \rightarrow \Diamond\Box\varphi$, we know $\mathcal{M}, \tau, x \models \Diamond\Box\varphi$, so there exists $x' > x$ with $\mathcal{M}, \tau, x' \models \Box\varphi$. Then $\mathcal{M}, \tau, y' \models \varphi$ for all $y' < x'$, so no element of B is less than x' and hence $c \geq x' > x$.

Let $y' < c$ be arbitrary. If $y' < x$, then $\mathcal{M}, \tau, y' \models \varphi$ by $\Box\varphi$. If $y' = x$, then $\mathcal{M}, \tau, y' \models \varphi$ since $x < x'$ and $\mathcal{M}, \tau, x' \models \Box\varphi$. If $x < y' < c$, it follows that $y' \notin B$, so $\mathcal{M}, \tau, y' \models \varphi$. Since $y' < c$ was arbitrary, we know that $\mathcal{M}, \tau, c \models \Box\varphi$. Given that $\mathcal{M}, \tau, c \models \Box\varphi \rightarrow \Diamond\Box\varphi$, we have $\mathcal{M}, \tau, c \models \Diamond\Box\varphi$, and so there exists $z > c$ with $\mathcal{M}, \tau, z \models \Box\varphi$. Since $c = \inf(B)$, there exists $b \in B$ with $c \leq b < z$. Then $\mathcal{M}, \tau, b \not\models \varphi$ with $b < z$, contradicting $\mathcal{M}, \tau, z \models \Box\varphi$. Therefore $\mathcal{M}, \tau, x \models \Box\varphi$. Since $\mathcal{F}$, $\mathcal{M}$, τ , and x were arbitrary, we may conclude that $\Delta(\Box\varphi \rightarrow \Diamond\Box\varphi) \rightarrow (\Box\varphi \rightarrow \Box\varphi)$ is valid over every frame $\mathcal{F} = \langle W, \mathcal{D}, \Rightarrow \rangle$.

($\Rightarrow$) Suppose $\mathcal{D}$ is not COMPLETE. Then there exists a nonempty $S \subseteq D$ bounded above with no least upper bound. Let $L = \{x \in D : x < y \text{ for some } y \in S\}$ and

$U = \{x \in D : y \leq x \text{ for all } y \in S\}$. Since $\mathcal{D}$ is totally ordered, L and U partition D with every element of L strictly less than every element of U . Since S has no least upper bound, L has no maximum and U has no minimum. Let $W = \{w_0, w_1\}$ with $w \Rightarrow_d w'$ for all $w, w' \in W$ and $d \in D$, let $\mathcal{F}' = \langle W, \mathcal{D}, \Rightarrow \rangle$, and define $\mathcal{M} = \langle W, \mathcal{D}, \Rightarrow, |\cdot| \rangle$ with $|p| = \{w_0\}$ where $\tau(x) = w_0$ for $x \in L$ and $\tau(x) = w_1$ for $x \in U$. Since $\Rightarrow$ is universal, $\tau \in H_{\mathcal{F}'}$.



Given [D22](#), it follows that $\mathcal{M}, \tau, x \models p$ if and only if $x \in L$. Pick any $x \in L$. Since L is downward closed, $\mathcal{M}, \tau, x \models \Box p$. Since U is nonempty and every $y \in U$ satisfies $y > x$ and $\tau(y) = w_1 \notin |p|$, we have $\mathcal{M}, \tau, x \not\models \Box p$.

It remains to verify $\mathcal{M}, \tau, x \models \Delta(\Box p \rightarrow \Diamond \Box p)$. Let $y \in D$ be arbitrary. If $y \in U$, then since U has no minimum, there exists $y' \in U$ with $y' < y$, and $\tau(y') = w_1 \notin |p|$, so $\mathcal{M}, \tau, y \not\models \Box p$ and the conditional holds vacuously. If $y \in L$, then $\mathcal{M}, \tau, y \models \Box p$ since L is downward closed. Since L has no maximum, there exists $y' \in L$ with $y' > y$. Every $z < y'$ lies in L by downward closure, so $\mathcal{M}, \tau, y' \models \Box p$, giving $\mathcal{M}, \tau, y \models \Diamond \Box p$. Since y was arbitrary, $\mathcal{M}, \tau, x \models \Delta(\Box p \rightarrow \Diamond \Box p)$. Since $\mathcal{M}, \tau, x \models \Box p$ but $\mathcal{M}, \tau, x \not\models \Box \Box p$, we have $\mathcal{M}, \tau, x \not\models \Box p \rightarrow \Box \Box p$, and so $\not\models_{\mathcal{F}'} \Delta(\Box \varphi \rightarrow \Diamond \Box \varphi) \rightarrow (\Box \varphi \rightarrow \Box \Box \varphi)$, and the axiom is not valid over every frame with temporal order $\mathcal{D}$. $\square$

L9 *If $\mathcal{F}$ is DETERMINISTIC, then $\langle \tau \rangle_x = \{\tau\}$ for all $\tau \in H_{\mathcal{F}}$ and $x \in D$. The converse fails: a frame with no task transitions of duration strictly between 0 and some fixed $d > 0$ can satisfy $\langle \tau \rangle_x = \{\tau\}$ everywhere without being DETERMINISTIC, since no possible world can then have a domain containing two times less than d apart.*

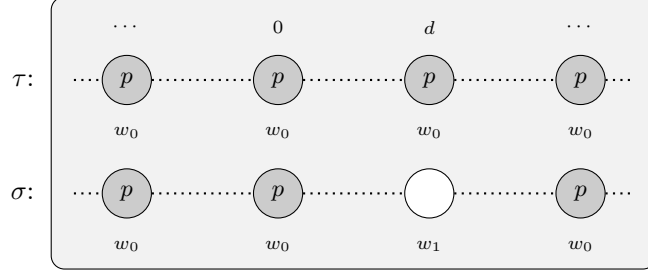
Proof. Suppose $\mathcal{F}$ is DETERMINISTIC. Since $\tau(x) = \tau(x)$, we have $\tau \in \langle \tau \rangle_x$. For the reverse inclusion, let $\sigma \in \langle \tau \rangle_x$ so that $\sigma(x) = \tau(x)$. For any $y \in \text{dom}(\tau)$, since τ is a possible world, $\tau(x) \Rightarrow_{y-x} \tau(y)$ by [D20](#). Since σ is a possible world, $\sigma(x) \Rightarrow_{y-x} \sigma(y)$ by [D20](#), and so $\tau(x) \Rightarrow_{y-x} \sigma(y)$. By DETERMINISTIC ([D24](#)), $\tau(y) = \sigma(y)$. Since y was arbitrary, we may conclude that $\sigma = \tau$ as desired. $\square$

T11 *$\varphi \rightarrow \Box \varphi$ is valid over every deterministic frame $\mathcal{F} = \langle W, \mathcal{D}, \Rightarrow \rangle$, but is invalid over some non-deterministic frame.*

Proof. Let $\mathcal{F} = \langle W, \mathcal{D}, \Rightarrow \rangle$ be DETERMINISTIC. Let $\mathcal{M} = \langle W, \mathcal{D}, \Rightarrow, |\cdot| \rangle$ be a model over $\mathcal{F}$ with $\tau \in H_{\mathcal{F}}$ and $x \in D$ where $\mathcal{M}, \tau, x \models \varphi$. By [L9](#), $\langle \tau \rangle_x = \{\tau\}$, so $\mathcal{M}, \sigma, x \models \varphi$

for all $\sigma \in \langle \tau \rangle_x$. Therefore $\mathcal{M}, \tau, x \models \Box \varphi$. Since $\mathcal{M}$, τ , and x were chosen arbitrarily, $\models_{\mathcal{F}} \varphi \rightarrow \Box \varphi$ for every deterministic frame $\mathcal{F}$.

Let $d \in D$ with $d > 0$. Letting $W = \{w_0, w_1\}$ with $w \Rightarrow_d w'$ for all $w, w' \in W$ and $d \in D$, let $\mathcal{F}' = \langle W, \mathcal{D}, \Rightarrow \rangle$, and define $\mathcal{M} = \langle W, \mathcal{D}, \Rightarrow, |\cdot| \rangle$ with $|p| = \{w_0\}$ and $\tau, \sigma : D \rightarrow W$ by $\tau(y) = w_0$ for all $y \in D$, and $\sigma(y) = w_1$ if $y = d$ and $\sigma(y) = w_0$ otherwise. Since $\Rightarrow$ relates every pair of states at every duration, $\mathcal{F}'$ is not DETERMINISTIC, and both τ and σ are in $H_{\mathcal{F}'}$.



Since $\sigma(0) = w_0 = \tau(0)$, we have $\sigma \in \langle \tau \rangle_0$. Since $\tau(y) = w_0 \in |p|$ for all $y > 0$, we have $\mathcal{M}, \tau, 0 \models \Box p$ by **D22**. However, $\sigma(d) = w_1 \notin |p|$ and $d > 0$, so $\mathcal{M}, \sigma, 0 \not\models \Box p$ by **D22**. Since $\sigma \in \langle \tau \rangle_0$ and $\mathcal{M}, \sigma, 0 \not\models \Box p$, we have $\mathcal{M}, \tau, 0 \not\models \Box \Box p$. Thus we may conclude that $\mathcal{M}, \tau, 0 \not\models \Box p \rightarrow \Box \Box p$, and so $\not\models_{\mathcal{F}'} \varphi \rightarrow \Box \varphi$ more generally, giving a non-deterministic frame over which the schema is invalid. $\square$

T12 $\uparrow_T^1 \Box \uparrow_T^2 \downarrow_T^1 (\Box \downarrow_T^2 \neg \varphi \vee \Box \downarrow_T^2 \varphi)$ is valid over every deterministic frame, and invalid over some non-deterministic frame.

Proof. We begin by observing that the biconditionals below follow from **D22**:

$$\begin{aligned}
\mathcal{M}, \tau, x, \vec{v} &\models \uparrow_T^1 \Box \uparrow_T^2 \downarrow_T^1 (\Box \downarrow_T^2 \neg \varphi \vee \Box \downarrow_T^2 \varphi) & (*) \\
&\Leftrightarrow \mathcal{M}, \tau, x, \vec{v}_{[x/v_1]} \models \Box \uparrow_T^2 \downarrow_T^1 (\Box \downarrow_T^2 \neg \varphi \vee \Box \downarrow_T^2 \varphi) \\
&\Leftrightarrow \mathcal{M}, \tau, y, \vec{v}_{[x/v_1]} \models \uparrow_T^2 \downarrow_T^1 (\Box \downarrow_T^2 \neg \varphi \vee \Box \downarrow_T^2 \varphi) \text{ for all } y > x \\
&\Leftrightarrow \mathcal{M}, \tau, y, \vec{v}_{[x/v_1][y/v_2]} \models \downarrow_T^1 (\Box \downarrow_T^2 \neg \varphi \vee \Box \downarrow_T^2 \varphi) \text{ for all } y > x \\
&\Leftrightarrow \mathcal{M}, \tau, x, \vec{v}_{[x/v_1][y/v_2]} \models \Box \downarrow_T^2 \neg \varphi \vee \Box \downarrow_T^2 \varphi \text{ for all } y > x.
\end{aligned}$$

Let $\mathcal{F} = \langle W, \mathcal{D}, \Rightarrow \rangle$ be DETERMINISTIC. Choose an arbitrary world $\tau \in H_{\mathcal{F}}$, times $x, y \in D$ where $x < y$, and stored times $\vec{v}$ in a model $\mathcal{M} = \langle W, \mathcal{D}, \Rightarrow, |\cdot| \rangle$ over $\mathcal{F}$. By **L9**, it follows that $\langle \tau \rangle_x = \{\tau\}$. If $\mathcal{M}, \tau, y, \vec{v}_{[x/v_1][y/v_2]} \models \varphi$, then $\mathcal{M}, \tau, x, \vec{v}_{[x/v_1][y/v_2]} \models \Box \downarrow_T^2 \varphi$ by **D22** and **T11**. If $\mathcal{M}, \tau, y, \vec{v}_{[x/v_1][y/v_2]} \not\models \varphi$, then $\mathcal{M}, \tau, x, \vec{v}_{[x/v_1][y/v_2]} \models \Box \downarrow_T^2 \neg \varphi$ by the same reasoning. Thus $\mathcal{M}, \tau, x, \vec{v}_{[x/v_1][y/v_2]} \models \Box \downarrow_T^2 \neg \varphi \vee \Box \downarrow_T^2 \varphi$ by **D22**. Given the biconditionals (*), we conclude this direction of the proof.

By the same non-deterministic frame $\mathcal{F}'$ and countermodel presented in **T11**, we have $\sigma(0) = w_0 = \tau(0)$ where $\sigma, \tau \in \langle \tau \rangle_0$. Since $\tau(d) = w_0 \in |p|$, we know that

$\mathcal{M}, \tau, d, \vec{v}_{[0/v_1][d/v_2]} \models p$, and so we may draw the following conclusions by **D22**:

$$\mathcal{M}, \tau, d, \vec{v}_{[0/v_1][d/v_2]} \models p \Leftrightarrow \mathcal{M}, \tau, 0, \vec{v}_{[0/v_1][d/v_2]} \models \downarrow_T^2 p \quad (1)$$

$$\mathcal{M}, \tau, d, \vec{v}_{[0/v_1][d/v_2]} \not\models \neg p \Leftrightarrow \mathcal{M}, \tau, 0, \vec{v}_{[0/v_1][d/v_2]} \not\models \downarrow_T^2 \neg p \quad (2)$$

Since the conditions on the left are equivalent, all four sides of the biconditionals above hold. Given that $\sigma(d) = w_1 \notin |p|$ we know that $\mathcal{M}, \sigma, d, \vec{v}_{[0/v_1][d/v_2]} \not\models p$ and so we may draw the following conclusions where again all four conditions are equivalent:

$$\mathcal{M}, \sigma, d, \vec{v}_{[0/v_1][d/v_2]} \not\models p \Leftrightarrow \mathcal{M}, \sigma, 0, \vec{v}_{[0/v_1][d/v_2]} \not\models \downarrow_T^2 p \quad (3)$$

$$\mathcal{M}, \sigma, d, \vec{v}_{[0/v_1][d/v_2]} \models \neg p \Leftrightarrow \mathcal{M}, \sigma, 0, \vec{v}_{[0/v_1][d/v_2]} \models \downarrow_T^2 \neg p \quad (4)$$

Since $\tau, \sigma \in \langle \tau \rangle_0$, it follows from (3) that $\mathcal{M}, \tau, 0, \vec{v}_{[0/v_1][d/v_2]} \not\models \Box \downarrow_T^2 p$, witnessed by σ , and it follows from (2) that $\mathcal{M}, \tau, 0, \vec{v}_{[0/v_1][d/v_2]} \not\models \Box \downarrow_T^2 \neg p$, witnessed by τ itself. Thus $\mathcal{M}, \tau, 0, \vec{v}_{[0/v_1][d/v_2]} \not\models \Box \downarrow_T^2 \neg p \vee \Box \downarrow_T^2 p$. By (*) above, we may conclude the proof. $\square$

D26 Given a frame $\mathcal{F} = \langle W, \mathcal{D}, \Rightarrow \rangle$, define:

Basic Opens: $(w)_x := \{u \in W : w \Rightarrow_y u \text{ for some } y \in D \text{ with } |y| < x\}$ for each $w \in W$ and $x > 0$.

Basis: $B_{\mathcal{F}} := \{(w)_x : w \in W \text{ and } x \in D \text{ with } x > 0\}$.

Topology: $\mathcal{T}_{\mathcal{F}} := \langle W, \mathcal{O}_{\mathcal{F}} \rangle$ where $\mathcal{O}_{\mathcal{F}}$ is the result of closing $B_{\mathcal{F}}$ under arbitrary union and finite intersection.

Discrete: A topology is *discrete* just in case every subset of W is open, and *non-discrete* otherwise.

Closure: $\bar{S} := \{w \in W : O \cap S \neq \emptyset \text{ for every open } O \in \mathcal{T}_{\mathcal{F}} \text{ where } w \in O\}$ for $S \subseteq W$.

R0: A topology is *R0* just in case $w \in \overline{\{u\}}$ iff $u \in \overline{\{w\}}$ for all $w, u \in W$.

T13 $\mathcal{T}_{\mathcal{F}}$ is *R0* for every frame $\mathcal{F}$.

Proof. Let $\mathcal{F} = \langle W, \mathcal{D}, \Rightarrow \rangle$ be a frame and $w, u \in W$. It suffices to test membership in $\overline{\{u\}}$ against the basic opens $(w)_x$ centered at w itself. Given any basic open $(v)_z \ni w$, say via $v \Rightarrow_y w$ with $|y| < z$, *Compositionality* gives $(w)_{z-|y|} \subseteq (v)_z$, since any $r \in (w)_{z-|y|}$ has $w \Rightarrow_{y'} r$ with $|y'| < z - |y|$ and so $v \Rightarrow_{y+y'} r$ with $|y+y'| \leq |y| + |y'| < z$. Hence every open containing w contains some $(w)_x$, and so it is enough to consider:

$$\begin{aligned} w \in \overline{\{u\}} &\Leftrightarrow u \in (w)_x \text{ for all } x > 0 \\ &\Leftrightarrow \text{for every } x > 0 \text{ there exists } y \text{ with } |y| < x \text{ such that } w \Rightarrow_y u. \end{aligned}$$

By **D19**, *Reflection* yields $u \Rightarrow_{-y} w$ where $|-y| = |y| < x$. It follows that $w \in \overline{\{u\}}$ just in case $u \in \overline{\{w\}}$, and so $\mathcal{T}_{\mathcal{F}}$ is R0 as desired. $\square$

T14 $\mathcal{T}_{\mathcal{F}}$ is discrete if and only if for every $w \in W$ there exists $x > 0$ such that $w \Rightarrow_y u$ with $|y| < x$ only when $u = w$.

Proof. ($\Rightarrow$) Suppose $\mathcal{T}_{\mathcal{F}}$ is discrete and let $w \in W$. Then $\{w\} \in \mathcal{O}_{\mathcal{F}}$, and so it follows that $\{w\} = \bigcup_{j \in J} O_j$ where each O_j is a finite intersection of elements of $B_{\mathcal{F}}$. Since $w \in \{w\}$, there exists $j \in J$ with $w \in O_j$. But $O_j \subseteq \{w\}$, so $O_j = \{w\}$. Writing $O_j = \bigcap_{k=1}^n (w_k)_{x_k}$, Nullity yields $w_k \in (w_k)_{x_k}$ for each k , and so $w_k \in \{w\}$, whence $w_k = w$. By nestedness— $(w)_x \subseteq (w)_y$ whenever $x \leq y$ —the intersection reduces to $(w)_{x_{\min}} = \{w\}$ where $x_{\min} := \min\{x_1, \dots, x_n\}$. Hence there exists $x > 0$ (namely $x_{\min}$) such that $w \Rightarrow_y u$ with $|y| < x$ only when $u = w$.

($\Leftarrow$) Suppose the antecedent holds and let $w \in W$. By assumption there exists $x > 0$ with $(w)_x = \{w\}$. Thus $\{w\}$ is a basic open, and since w was arbitrary, every singleton is open. Therefore every subset of W is open, and $\mathcal{T}_{\mathcal{F}}$ is discrete. $\square$

5.4 Soundness

The purely modal axioms **MK**, **MT**, and **M5** are sound by standard reasoning since $\Box$ quantifies universally over all possible worlds $H_{\mathcal{F}}$ without accessibility restriction, validating an S5 modal logic. Similarly, the temporal axioms **TK**, **TL**, **T4**, **TA**, and **TB** are sound given that $\mathcal{D}$ is a totally ordered abelian group where $\mathcal{D}$ is unbounded. Concise semantic proofs for representative axioms from each group are provided below, and the full soundness proof has been formalized in the [Lean 4 repository](#) for this paper. I will nevertheless present validity proofs for a number of characteristic axioms including the bimodal interaction axiom **MF** and the exchange rule **TD**.

D27 A conclusion φ is a *logical consequence* of a set of premises Γ —written $\Gamma \models \varphi$ —just in case for all models $\mathcal{M}$, possible worlds $\tau \in H_{\mathcal{F}}$, and times $x \in T$, if $\mathcal{M}, \tau, x \models \gamma$ for all premises $\gamma \in \Gamma$, then $\mathcal{M}, \tau, x \models \varphi$. A sentence φ is *valid* just in case $\models \varphi$.

D28 The derivation relation $\vdash$ for **TM** is the smallest relation closed under the axioms and rules for **TM** as presented in [§3.2](#).

D29 The proof system **TM** is *sound* with respect to the task semantics just in case for every sentence φ and set of sentences Γ in $\mathcal{L}$, $\Gamma \models \varphi$ whenever $\Gamma \vdash \varphi$.

T15 $\models \Diamond \Box \varphi \rightarrow \Box \varphi$.

Proof. Suppose $\mathcal{M}, \tau, x \models \Diamond \Box \varphi$. By **D22**, there exists $\sigma \in H_{\mathcal{F}}$ such that $\mathcal{M}, \sigma, x \models \Box \varphi$. Then $\mathcal{M}, \rho, x \models \varphi$ for all $\rho \in H_{\mathcal{F}}$. But this is exactly the condition for $\mathcal{M}, \tau, x \models \Box \varphi$, since the quantification ranges over $H_{\mathcal{F}}$. Thus $\models \Diamond \Box \varphi \rightarrow \Box \varphi$. $\square$

L10 Let $n : D \rightarrow D$ be defined by $n(x) := -x$, let $\mathcal{F}^- = \langle W, \mathcal{D}, \Rightarrow^- \rangle$ where $w \Rightarrow_x^- u := w \Rightarrow_{-x} u$, and let $\tau^- = \tau \circ n$ for each $\tau \in H_{\mathcal{F}}$. Then $\mathcal{F}^-$ is a task frame and $\tau \mapsto \tau^-$ is a bijection $H_{\mathcal{F}} \rightarrow H_{\mathcal{F}^-}$, and for all well-formed sentences φ of $\mathcal{L}$, models $\mathcal{M} = \langle W, \mathcal{D}, \Rightarrow, |\cdot| \rangle$ over $\mathcal{F}$ and $\mathcal{M}^- = \langle W, \mathcal{D}, \Rightarrow^-, |\cdot| \rangle$ over $\mathcal{F}^-$, possible worlds $\tau \in H_{\mathcal{F}}$, and times $x \in D$:

$$\mathcal{M}, \tau, x \models \varphi \Leftrightarrow \mathcal{M}^-, \tau^-, n(x) \models \varphi_{\langle P|F \rangle}.$$

Proof. Since $\mathcal{D}$ is a totally ordered abelian group, it follows that n is an order-reversing automorphism satisfying the following:

$$n(x + y) = -(x + y) = (-x) + (-y) = n(x) + n(y) \quad (\dagger)$$

$$x < y \Leftrightarrow -y < -x \Leftrightarrow n(y) < n(x) \quad (\ddagger)$$

Since $\mathcal{F}^-$ is a task frame, *Nullity* and *Reflection* transfer immediately from $\Rightarrow$ to $\Rightarrow^-$, and *Compositionality* holds since $-x + -y = -(x + y)$ by $(\dagger)$. Given that $\tau \in H_{\mathcal{F}}$ and $a, b \in \text{dom}(\tau^-) = n(\text{dom}(\tau))$, we have $\tau^-(a) \Rightarrow_{b-a}^- \tau^-(b)$ just in case $\tau(-a) \Rightarrow_{a-b} \tau(-b)$, which holds by **D20** applied to τ at $-a, -b \in \text{dom}(\tau)$; thus $\tau^- \in H_{\mathcal{F}^-}$. Since $n \circ n = \text{id}$ so that $(\mathcal{F}^-)^- = \mathcal{F}$, the same argument applied to $\mathcal{F}^-$ shows that $\sigma \in H_{\mathcal{F}^-}$ implies $\sigma^- \in H_{\mathcal{F}}$, and so $\tau \mapsto \tau^-$ is a bijection $H_{\mathcal{F}} \rightarrow H_{\mathcal{F}^-}$ with itself as inverse. The proof of the displayed biconditional is by induction on the complexity of φ .

Base Case (p_i): Since $(p_i)_{\langle P|F \rangle} = p_i$ and $\mathcal{M}, \mathcal{M}^-$ share the interpretation $|\cdot|$, the biconditional follows from **D22**:

$$\begin{aligned} \mathcal{M}, \tau, x \models p_i &\Leftrightarrow x \in \text{dom}(\tau) \text{ and } \tau(x) \in |p_i| \\ &\Leftrightarrow n(x) \in \text{dom}(\tau^-) \text{ and } \tau^-(n(x)) \in |p_i| \\ &\Leftrightarrow \mathcal{M}^-, \tau^-, n(x) \models p_i \end{aligned}$$

The second biconditional holds since $n(x) \in \text{dom}(\tau^-)$ just in case $x \in \text{dom}(\tau)$, where we also know that $\tau^-(n(x)) = \tau(n(n(x))) = \tau(x)$.

Base Case ($\perp$): $\mathcal{M}, \tau, x \not\models \perp$ and $\mathcal{M}^-, \tau^-, n(x) \not\models \perp$ by **D22**.

Inductive Case ($\varphi \rightarrow \psi$): Since $(\varphi \rightarrow \psi)_{\langle P|F \rangle} = \varphi_{\langle P|F \rangle} \rightarrow \psi_{\langle P|F \rangle}$, the result follows from the inductive hypothesis and **D22**.

Inductive Case ($\Box\varphi$): Since $(\Box\varphi)_{\langle P|F \rangle} = \Box\varphi_{\langle P|F \rangle}$, the biconditional follows by **D22**:

$$\begin{aligned} \mathcal{M}, \tau, x \models \Box\varphi &\Leftrightarrow \mathcal{M}, \sigma, x \models \varphi \text{ for all } \sigma \in H_{\mathcal{F}} \\ &\Leftrightarrow \mathcal{M}^-, \sigma^-, n(x) \models \varphi_{\langle P|F \rangle} \text{ for all } \sigma \in H_{\mathcal{F}} \\ &\Leftrightarrow \mathcal{M}^-, \rho, n(x) \models \varphi_{\langle P|F \rangle} \text{ for all } \rho \in H_{\mathcal{F}^-} \\ &\Leftrightarrow \mathcal{M}^-, \tau^-, n(x) \models \Box\varphi_{\langle P|F \rangle} \end{aligned}$$

The second biconditional holds by the inductive hypothesis, the third since $\sigma \mapsto \sigma^-$ is a bijection $H_{\mathcal{F}} \rightarrow H_{\mathcal{F}^-}$, and the fourth by **D22** applied to $\mathcal{M}^-$ over $\mathcal{F}^-$.

Inductive Case ($\Box\varphi$): Since $(\Box\varphi)_{\langle P|F \rangle} = \Box\varphi_{\langle P|F \rangle}$, the biconditional follows by **D22**:

$$\begin{aligned} \mathcal{M}, \tau, x \models \Box\varphi &\Leftrightarrow \mathcal{M}, \tau, y \models \varphi \text{ for all } y \in D \text{ where } x < y \\ &\Leftrightarrow \mathcal{M}^-, \tau^-, n(y) \models \varphi_{\langle P|F \rangle} \text{ for all } y \in D \text{ where } x < y \\ &\Leftrightarrow \mathcal{M}^-, \tau^-, z \models \varphi_{\langle P|F \rangle} \text{ for all } z \in D \text{ where } z < n(x) \\ &\Leftrightarrow \mathcal{M}^-, \tau^-, n(x) \models \Box\varphi_{\langle P|F \rangle} \end{aligned}$$

The second biconditional holds by the inductive hypothesis. The third holds by ($\ddagger$), and so substituting $z = n(y)$ transforms the domain of quantification.

Inductive Case ($\Box\varphi$): By parity of reasoning with the $\Box$ case, replacing $y > x$ with $y < x$ and applying ($\ddagger$) in the opposite direction. $\square$

T16 If $\models \varphi$, then $\models \varphi_{\langle P|F \rangle}$.

Proof. Suppose $\models \varphi$ and let $\mathcal{M} = \langle W, \mathcal{D}, \Rightarrow, |\cdot| \rangle$ be a model of $\mathcal{L}$ defined over the task frame $\mathcal{F} = \langle W, \mathcal{D}, \Rightarrow \rangle$, where $\sigma \in H_{\mathcal{F}}$ and $y \in D$ be arbitrary. Let $\mathcal{F}^- = \langle W, \mathcal{D}, \Rightarrow^- \rangle$ and $\mathcal{M}^- = \langle W, \mathcal{D}, \Rightarrow^-, |\cdot| \rangle$ be as in **L10**. Since $\mathcal{F}^-$ is itself a task frame, $\mathcal{M}^-$ is a model of $\mathcal{L}$, and since $\models \varphi$ holds over every model, we have $\mathcal{M}^-, \sigma^-, n(y) \models \varphi$, where $\sigma^- \in H_{\mathcal{F}^-}$ by the bijection established in **L10**.

Applying **L10** to $\mathcal{M}^-$ over $\mathcal{F}^-$ with $\tau = \sigma^-$ and $x = n(y)$ yields:

$$(\mathcal{M}^-)^-, (\sigma^-)^-, n(n(y)) \models \varphi_{\langle P|F \rangle}.$$

Since $(\mathcal{F}^-)^- = \mathcal{F}$ so that $(\mathcal{M}^-)^- = \mathcal{M}$, and since $(\sigma^-)^- = \sigma$ and $n(n(y)) = y$, we conclude $\mathcal{M}, \sigma, y \models \varphi_{\langle P|F \rangle}$ as desired. Equivalently, $\models \varphi_{\langle P|F \rangle}$. $\square$

T17 $\models \varphi \rightarrow \Box\Diamond\varphi$.

Proof. Suppose $\mathcal{M}, \tau, x \models \varphi$. To show $\mathcal{M}, \tau, x \models \Box\Diamond\varphi$, let $y > x$ be arbitrary. Thus $\mathcal{M}, \tau, y \models \Diamond\varphi$. Since $y > x$ was arbitrary, $\mathcal{M}, \tau, x \models \Box\Diamond\varphi$. Thus $\models \varphi \rightarrow \Box\Diamond\varphi$. $\square$

T18 $\models (\Diamond\varphi \wedge \Diamond\psi) \rightarrow [\Diamond(\Diamond\varphi \wedge \psi) \vee \Diamond(\varphi \wedge \psi) \vee \Diamond(\varphi \wedge \Diamond\psi)]$.

Proof. Suppose $\mathcal{M}, \tau, x \models \Diamond\varphi$ and $\mathcal{M}, \tau, x \models \Diamond\psi$. By **D22**, there exist $y > x$ and $z > x$ with $\mathcal{M}, \tau, y \models \varphi$ and $\mathcal{M}, \tau, z \models \psi$. Since $\mathcal{D}$ is totally ordered, either $y < z$, $y = z$, or $z < y$. If $y < z$, then $\mathcal{M}, \tau, y \models \varphi \wedge \Diamond\psi$, giving $\mathcal{M}, \tau, x \models \Diamond(\varphi \wedge \Diamond\psi)$. If $y = z$, then $\mathcal{M}, \tau, y \models \varphi \wedge \psi$, so $\mathcal{M}, \tau, x \models \Diamond(\varphi \wedge \psi)$. If $z < y$, then $\mathcal{M}, \tau, z \models \Diamond\varphi \wedge \psi$, giving $\mathcal{M}, \tau, x \models \Diamond(\Diamond\varphi \wedge \psi)$. Thus $\models (\Diamond\varphi \wedge \Diamond\psi) \rightarrow [\Diamond(\Diamond\varphi \wedge \psi) \vee \Diamond(\varphi \wedge \psi) \vee \Diamond(\varphi \wedge \Diamond\psi)]$. $\square$

T19 $\models \Box\varphi \rightarrow \Box\Box\varphi$.

Proof. Suppose $\mathcal{M}, \tau, x \models \Box\varphi$. By **D22**, $\mathcal{M}, \sigma, x \models \varphi$ for all $\sigma \in H_{\mathcal{F}}$. In order to show $\mathcal{M}, \tau, x \models \Box\Box\varphi$, we must prove that $\mathcal{M}, \sigma, x \models \Box\varphi$ for all $\sigma \in H_{\mathcal{F}}$.

Let $\sigma \in H_{\mathcal{F}}$ be arbitrary and consider any $y > x$. By **L7**, there is some $\rho \in H_{\mathcal{F}}$ such that $\sigma \approx_y^x \rho$ is witnessed by the time-shift function $\bar{a}(z) = z - y + x$. By **L8**, $\mathcal{M}, \sigma, y \models \varphi$

if and only if $\mathcal{M}, \rho, x \models \varphi$. Since $\mathcal{M}, \tau, x \models \Box\varphi$, we have $\mathcal{M}, \rho, x \models \varphi$. Therefore $\mathcal{M}, \sigma, y \models \varphi$ by time-shift invariance. Since $y > x$ was arbitrary, $\mathcal{M}, \sigma, x \models \Box_F\varphi$. Since σ was arbitrary, $\mathcal{M}, \tau, x \models \Box\Box_F\varphi$. Thus $\models \Box\varphi \rightarrow \Box\Box_F\varphi$. $\square$

T20 (Soundness) *If $\vdash \varphi$, then $\models \varphi$.*

Proof. By induction on derivations. The propositional tautologies are valid, and **MP** preserves validity by standard arguments. The S5 axioms **MK**, **MT**, and **M5** are valid given that $\Box$ quantifies universally over $H_{\mathcal{F}}$, and **MN** preserves validity. The temporal axioms **TK**, **TL**, **T4**, **TA**, and **TB** are valid given that $\mathcal{D}$ is a totally ordered abelian group, and **TD** preserves validity since there is an order-reversing automorphism $x \mapsto -x$ on $\mathcal{D}$. Representative proofs are given in **T15**, **T16**, **T17**, and **T18** above. The bimodal axiom **MF** is shown to be valid in **T19**. The proof of soundness otherwise proceeds as usual.⁷² $\square$

C2 *The perpetuity principles **P1** – **P6** are all valid.*

Proof. Follows immediately from **T20** given the derivations of **P1** – **P6** in §3.2. $\square$

5.5 Proof Theory

This section presents the proof system **TM**⁺ for the extended language $\mathcal{L}^+$ in order to state the result that **TM**⁺ is a conservative extension of **TM** which is implemented in the Lean 4 [repository](#) for this paper. All definitions are restated for convenience.

D30 The language $\mathcal{L}^+ := \langle \mathbb{L}, \perp, \rightarrow, \Box, \triangleleft, \triangleright \rangle$ where $\mathbb{L} := \{p_i : i \in \mathbb{N}\}$ is a countable set of sentence letters as before where the remaining symbols denote falsity, material implication, the metaphysical necessity operator, the since operator, and the until operator, respectively. Well-formed sentences of $\mathcal{L}^+$ are defined by:

$$\varphi, \psi ::= p_i \mid \perp \mid \varphi \rightarrow \psi \mid \Box\varphi \mid \varphi \triangleleft \psi \mid \varphi \triangleright \psi.$$

D31 The *models* of $\mathcal{L}^+$ are defined in **D21**, where *truth in a model* $\mathcal{M}$ at $\tau \in H_{\mathcal{F}}$ and $x \in D$ extends the semantics **D22** with the following clauses:⁷³

- ($\triangleleft$) $\mathcal{M}, \tau, x \models \varphi \triangleleft \psi$ iff $\mathcal{M}, \tau, z \models \psi$ for some time $z < x$ where $\mathcal{M}, \tau, y \models \varphi$ for all $y \in D$ with $z < y < x$.
- ($\triangleright$) $\mathcal{M}, \tau, x \models \varphi \triangleright \psi$ iff $\mathcal{M}, \tau, z \models \psi$ for some time $z > x$ where $\mathcal{M}, \tau, y \models \varphi$ for all $y \in D$ with $x < y < z$.

⁷²The repository <https://github.com/benbrastmckie/BimodalLogic> for this paper implements the soundness theorem for **TM** in Lean 4, providing formal verification for this result.

⁷³Although the axioms of **TM**⁺ are drawn from the Burgess-Xu (BX) system, the repository's **snce**/**until** constructors follow the Pnueli convention with the guard as the first argument and the event as the second: $\varphi \triangleleft \psi$ means ψ held at some past time with φ holding throughout the interval since, and $\varphi \triangleright \psi$ means ψ will hold at some future time with φ holding throughout the interval until ψ holds.

D32 The following operators are defined in $\mathcal{L}^+$:

Past: $\Diamond\varphi := \top \triangleleft \varphi$.

Always: $\Delta\varphi := \Box\varphi \wedge \varphi \wedge \Box\varphi$.

Future: $\Diamond\varphi := \top \triangleright \varphi$.

Sometimes: $\nabla\varphi := \Diamond\varphi \vee \varphi \vee \Diamond\varphi$.

Historical: $\Box\varphi := \neg\Diamond\neg\varphi$.

Next: $\odot\varphi := \perp \triangleright \varphi$.

Henceforth: $\Box\varphi := \neg\Diamond\neg\varphi$.

Previous: $\odot\varphi := \perp \triangleleft \varphi$.

T21 $\mathcal{M}, \tau, x \models \Box\varphi$ iff $\mathcal{M}, \tau, y \models \varphi$ for all $y \in D$ with $y < x$; and $\mathcal{M}, \tau, x \models \Box\varphi$ iff $\mathcal{M}, \tau, y \models \varphi$ for all $y \in D$ with $x < y$.

Proof. By **D32**, $\Box\varphi := \neg\Diamond\neg\varphi = \neg(\top \triangleleft \neg\varphi)$. By **D31**, $\mathcal{M}, \tau, x \models \top \triangleleft \neg\varphi$ iff there exists $z < x$ such that $\mathcal{M}, \tau, z \models \neg\varphi$ and $\mathcal{M}, \tau, y \models \top$ for all $y \in D$ with $z < y < x$. ($\Leftarrow$) If φ holds at every $y < x$, then no open interval (z, x) is throughout $\neg\varphi$, so $\top \triangleleft \neg\varphi$ is false, and so $\Box\varphi$ holds. ($\Rightarrow$) Contrapositive: if φ fails at some $z' < x$, then $\neg\varphi$ holds at z' , where $\top$ holds at all $y' \in D$ whatsoever, and so *a fortiori* whenever $z' < y' < x$. Thus $\mathcal{M}, \tau, x \models \top \triangleleft \neg\varphi$ by **D31**, and so $\Box\varphi$ then fails at x by **D32**.

By **D32**, $\Box\varphi := \neg\Diamond\neg\varphi = \neg(\top \triangleright \neg\varphi)$. By **D31**, $\mathcal{M}, \tau, x \models \top \triangleright \neg\varphi$ iff there exists $z > x$ such that $\mathcal{M}, \tau, z \models \neg\varphi$ and $\mathcal{M}, \tau, y \models \top$ for all $y \in D$ with $x < y < z$. ($\Leftarrow$) If φ holds at every $y > x$, then no open interval (x, z) is throughout $\neg\varphi$, so $\top \triangleright \neg\varphi$ is false, and so $\Box\varphi$ holds. ($\Rightarrow$) Contrapositive: if φ fails at some $z' > x$, then $\neg\varphi$ holds at z' , where $\top$ holds at all $y' \in D$ whatsoever, and so *a fortiori* whenever $x < y' < z'$. Thus $\mathcal{M}, \tau, x \models \top \triangleright \neg\varphi$ by **D31**, and so $\Box\varphi$ then fails at x by **D32**. $\square$

T22 Over DISCRETE frames, $\mathcal{M}, \tau, x \models \odot\varphi$ iff $\mathcal{M}, \tau, y \models \varphi$ for the immediate successor y of x ; and $\mathcal{M}, \tau, x \models \odot\varphi \leftrightarrow \perp$ when x has no immediate successor. The past dual holds for $\odot\varphi$.

Proof. By **D32**, $\odot\varphi := \perp \triangleright \varphi$. By **D31**, $\mathcal{M}, \tau, x \models \perp \triangleright \varphi$ iff there exists $z > x$ with $\mathcal{M}, \tau, z \models \varphi$ and $\mathcal{M}, \tau, y \models \perp$ for all $y \in D$ with $x < y < z$. Since $\perp$ is false everywhere, the interval (x, z) must contain no elements of D . Over DISCRETE frames (**D24**), the interval (x, z) is empty precisely when z is the immediate successor of x . Hence $\mathcal{M}, \tau, x \models \odot\varphi$ iff φ holds at the immediate successor of x ; when no immediate successor exists, the required empty interval cannot be witnessed and $\odot\varphi$ is equivalent to $\perp$. The proof for $\odot\varphi := \perp \triangleleft \varphi$ is similar. $\square$

D33 The **S5** Modal Logic is the smallest extension of *Classical Propositional Logic* CPL closed under the following rule schemata and metarule:

MK $\Box(\varphi \rightarrow \psi) \rightarrow (\Box\varphi \rightarrow \Box\psi)$.

MP $\varphi, \varphi \rightarrow \psi \vdash \psi$.

MT $\Box\varphi \rightarrow \varphi$.

MN If $\vdash \varphi$, then $\vdash \Box\varphi$.

M5 $\Diamond\Box\varphi \rightarrow \Box\varphi$.

D34 Letting $\varphi_{\langle s|u \rangle}$ denote the result of exchanging every occurrence of $\triangleleft$ with $\triangleright$ in φ (and vice versa), $\mathbf{BX}_b$ is the *Base Burgess–Xu Tense Logic* for $\triangleleft$ and $\triangleright$ where the past/since direction of each axiom follows from the future/until direction by $\triangleright$, and so only the future/until direction is stated:

TN If $\vdash \varphi$, then $\vdash \Box \varphi$.

TD If $\vdash \varphi$, then $\vdash \varphi_{\langle s|u \rangle}$.

The seriality and connectedness axioms, extending **TB** and **TL** of **TM**, are:

TB $\Diamond \top$.

CN $(\varphi \triangleright \psi) \wedge (\chi \triangleright \theta) \rightarrow [(\varphi \wedge \chi) \triangleright (\psi \wedge \theta) \vee (\varphi \wedge \chi) \triangleright (\psi \wedge \chi) \vee (\varphi \wedge \chi) \triangleright (\varphi \wedge \theta)]$.

TL $\Diamond \varphi \wedge \Diamond \psi \rightarrow [\Diamond(\varphi \wedge \psi) \vee \Diamond(\varphi \wedge \Diamond \psi) \vee \Diamond(\Diamond \varphi \wedge \psi)]$.

The primary axioms for $\triangleleft$ and $\triangleright$ are:

TA $\varphi \rightarrow \Box \Diamond \varphi$.

UC $\Box(\varphi \rightarrow \psi) \rightarrow ((\chi \triangleright \varphi) \rightarrow (\chi \triangleright \psi))$.

UE $(\varphi \triangleright \psi) \rightarrow \Diamond \psi$.

UF $(\varphi \triangleright \psi) \rightarrow (\varphi \wedge (\varphi \triangleright \psi)) \triangleright \psi$.

UT $\Diamond \varphi \rightarrow (\top \triangleright \varphi)$.

UG $\Box(\varphi \rightarrow \chi) \rightarrow ((\varphi \triangleright \psi) \rightarrow (\chi \triangleright \psi))$.

UI $\varphi \triangleright (\varphi \wedge (\varphi \triangleright \psi)) \rightarrow \varphi \triangleright \psi$.

SU $\theta \wedge (\varphi \triangleright \psi) \rightarrow \varphi \triangleright (\psi \wedge (\varphi \triangleleft \theta))$.

The uniformity axioms, which hold vacuously unless the order is discrete, are:

NP $\Box \top \rightarrow \Diamond \top$.

NA $\Box \top \rightarrow \Box \Diamond \top$.

NF $\Diamond \top \rightarrow \Box \Diamond \top$.

NB $\Diamond \top \rightarrow \Box \Diamond \top$.

D35 The *Discrete Burgess–Xu Tense Logic* $\mathbf{BX}_f$ is the smallest extension of the base logic $\mathbf{BX}_b$ to include all instances of the following axioms:

UZ $\Diamond \varphi \rightarrow (\neg \varphi \triangleright \varphi)$.

Z1 $\Box(\Box \varphi \rightarrow \varphi) \rightarrow (\Diamond \Box \varphi \rightarrow \Box \varphi)$.

Whereas **UZ** asserts that if φ holds at some future time, then there is a *nearest* future φ -time where $\neg \varphi$ throughout the intervening interval, **Z1** is a backward induction principle characteristic of successor-Archimedean frames.

D36 The *Dense Burgess–Xu Tense Logic* $\mathbf{BX}_d$ is the smallest extension of the base logic $\mathbf{BX}_b$ to include all instances of the following axioms:

DN $\Box \Box \varphi \rightarrow \Box \varphi$.

NN $\neg \Diamond \top$.

Whereas **DN** coincides with **DN** of **TM**, the axiom **NN** is specific to **TM**⁺ and asserts that there is no immediate successor.

D37 The *Complete Burgess–Xu Tense Logic* $\mathbf{BX}_c$ is the smallest extension of the base logic $\mathbf{BX}_b$ to include all instances of the following axioms:

$$\mathbf{CO} \quad \Delta(\Box\varphi \rightarrow \Diamond\Box\varphi) \rightarrow (\Box\varphi \rightarrow \Box\Box\varphi).$$

This axiom coincides with $\mathbf{CO}$ in $\mathbf{TM}$, though it is expressed in $\mathcal{L}^+$.⁷⁴

D38 The *Base Logic of Tense and Modality* $\mathbf{TM}^+$ for $\mathcal{L}^+$ is the smallest extension of $\mathbf{S5}$ and the base logic $\mathbf{BX}_b$ that includes the following *bimodal interaction* axiom:

$$\mathbf{MF} \quad \Box\varphi \rightarrow \Box\Box\varphi.$$

Similarly, the discrete $\mathbf{TM}_f^+$, dense $\mathbf{TM}_d^+$, and complete $\mathbf{TM}_c^+$ extensions of $\mathbf{TM}^+$ include the additional axioms that distinguish $\mathbf{BX}_f$, $\mathbf{BX}_d$, and $\mathbf{BX}_c$, respectively.

T23 (Conservative Extension) *The axioms and rules for $\mathcal{L}^+$ enrich expressivity without proving new $\mathcal{L}$ -theorems, and so $\mathbf{TM}^+$ is a conservative extension of $\mathbf{TM}$. More precisely, for any $\mathcal{L}$ -formula φ :*

$$\begin{array}{ll} \mathbf{CEB} & \mathbf{TM}^+ \vdash \varphi \Leftrightarrow \mathbf{TM} \vdash \varphi. \\ \mathbf{CED} & \mathbf{TM}_d^+ \vdash \varphi \Leftrightarrow \mathbf{TM}_d \vdash \varphi. \\ \mathbf{CEF} & \mathbf{TM}_f^+ \vdash \varphi \Leftrightarrow \mathbf{TM}_f \vdash \varphi. \\ \mathbf{CEC} & \mathbf{TM}_c^+ \vdash \varphi \Leftrightarrow \mathbf{TM}_c \vdash \varphi. \end{array}$$

Proof. The backwards direction follows from embedding $\mathcal{L}$ into $\mathcal{L}^+$, whereas the forward direction is established in the Lean 4 [repository](#). $\square$

C3 (Completeness) *$\mathbf{TM}$, $\mathbf{TM}_f$, $\mathbf{TM}_d$, $\mathbf{TM}_c$, and $\mathbf{TM}_{dc}$ are all complete, over the classes of all frames, discrete frames, dense frames, complete frames, and dense-and-complete frames, respectively.*

Proof. A $\mathcal{L}$ -formula has the same validity in $\mathcal{L}$ and $\mathcal{L}^+$, since the two languages share their models (**D31**). So if $\models \varphi$ for a $\mathcal{L}$ -formula φ , then φ is $\mathcal{L}^+$ -valid, whence $\mathbf{TM}^+ \vdash \varphi$ by the completeness of $\mathbf{TM}^+$, and so $\mathbf{TM} \vdash \varphi$ by **T23** (**CEB**). The completeness of $\mathbf{TM}$ thus rests entirely on that of its extension $\mathbf{TM}^+$. The same argument reduces the completeness of $\mathbf{TM}_f$, $\mathbf{TM}_d$, and $\mathbf{TM}_c$, via **CEF**, **CED**, and **CEC**, to that of $\mathbf{TM}_f^+$, $\mathbf{TM}_d^+$, and $\mathbf{TM}_c^+$ (established in the Lean 4 [repository](#)), and $\mathbf{TM}_{dc}$ to the same by combining **CED** and **CEC**. $\square$

C4 (Decidability) *$\mathbf{TM}$, $\mathbf{TM}_f$, $\mathbf{TM}_d$, $\mathbf{TM}_c$, and $\mathbf{TM}_{dc}$ are all decidable.*

Proof. Completeness together with a recursive axiomatization makes the theorems of each system recursively enumerable. The finite model property makes the non-theorems recursively enumerable as well, since every non-theorem then fails in some finite model, and finite models may be effectively enumerated and checked. A logic

⁷⁴Note that every Archimedean discrete temporal order is complete (Dedekind), so $\mathbf{TM}_{fc}^+ = \mathbf{TM}_f^+$ for Archimedean frames. The complete extension is substantive for non-Archimedean orders.

whose theorems and non-theorems are both recursively enumerable is decidable. The finite model property for **TM**, **TM_F**, **TM_D**, **TM_C**, and **TM_{DC}** is established in the Lean 4 [repository](#) for this paper, from which decidability follows. $\square$

Since derivations of **P1** – **P6** and **TF** are provided in §3.2, this section will conclude by deriving a number of additional interaction principles in **TM⁺**. I will begin by stating the following equivalences without proof.

$$\mathbf{P9} \quad \neg\Delta\varphi \leftrightarrow \nabla\neg\varphi.$$

$$\mathbf{P10} \quad \neg\nabla\varphi \leftrightarrow \Delta\neg\varphi.$$

Further derivations are provided in the [Lean 4 repository](#) for this paper.

$$\mathbf{P11} \quad \nabla\Diamond\varphi \rightarrow \Diamond\varphi.$$

Proof. Since $\nabla\Diamond\varphi := \Diamond\Diamond\varphi \vee \Diamond\varphi \vee \Diamond\Diamond\varphi$ where $\Diamond\varphi \rightarrow \Diamond\varphi$ is a theorem of classical propositional logic, it suffices to show $\Diamond\Diamond\varphi \rightarrow \Diamond\varphi$ and $\Diamond\Diamond\varphi \rightarrow \Diamond\varphi$.

By the derived theorem **TF** instantiated with $\neg\varphi$, we have $\Box\neg\varphi \rightarrow \mathsf{F}\Box\neg\varphi$, so $\Diamond\Diamond\varphi \rightarrow \Diamond\varphi$ by contraposition. By **TD** applied to **TF**, we have $\Box\neg\varphi \rightarrow \mathsf{P}\Box\neg\varphi$. It follows by contraposition that $\Diamond\Diamond\varphi \rightarrow \Diamond\varphi$ as desired. $\square$

$$\mathbf{P12} \quad \Delta\Diamond\varphi \rightarrow \Diamond\varphi.$$

Proof. Since $\Delta\Diamond\varphi := \mathsf{P}\Diamond\varphi \wedge \Diamond\varphi \wedge \mathsf{F}\Diamond\varphi$, conjunction elimination yields $\Delta\Diamond\varphi \rightarrow \Diamond\varphi$. $\square$

$$\mathbf{P13} \quad \nabla\Box\varphi \leftrightarrow \Box\varphi.$$

Proof. By **P6**, we have $\nabla\Box\varphi \rightarrow \Box\Delta\varphi$. Since $\Delta\varphi \rightarrow \varphi$ by propositional logic, we know by **MN** and **MK** that $\Box\Delta\varphi \rightarrow \Box\varphi$. Therefore $\nabla\Box\varphi \rightarrow \Box\varphi$. Since $\nabla\varphi := \Diamond\varphi \vee \varphi \vee \Diamond\varphi$, the reverse direction $\Box\varphi \rightarrow \nabla\Box\varphi$ follows by disjunction introduction. $\square$

$$\mathbf{P14} \quad \Delta\Box\varphi \leftrightarrow \Box\varphi.$$

Proof. Given the definition $\Delta\varphi := \mathsf{P}\varphi \wedge \varphi \wedge \mathsf{F}\varphi$, the forward direction $\Delta\Box\varphi \rightarrow \Box\varphi$ follows by conjunction elimination. For the reverse direction, we have $\Box\varphi \rightarrow \Box\varphi$ trivially, $\Box\varphi \rightarrow \mathsf{F}\Box\varphi$ by the derived theorem **TF**, and $\Box\varphi \rightarrow \mathsf{P}\Box\varphi$ by **TD** applied to **TF**. Therefore $\Box\varphi \rightarrow (\mathsf{P}\Box\varphi \wedge \Box\varphi \wedge \mathsf{F}\Box\varphi)$, which is $\Box\varphi \rightarrow \Delta\Box\varphi$ as desired. $\square$

$$\mathbf{P15} \quad \Box\Delta\varphi \leftrightarrow \Delta\Box\varphi.$$

Proof. Since $\Delta\varphi \rightarrow \varphi$ by definition, **MN** and **MK** yield $\Box\Delta\varphi \rightarrow \Box\varphi$. By **P14**, $\Box\varphi \rightarrow \Delta\Box\varphi$, and so $\Box\Delta\varphi \rightarrow \Delta\Box\varphi$, establishing the forward direction.

Since **P14** gives $\Delta\Box\varphi \rightarrow \Box\varphi$, we know $\Box\varphi \rightarrow \Box\Delta\varphi$ by **P3**, and so $\Delta\Box\varphi \rightarrow \Box\Delta\varphi$, establishing the reverse. $\square$

$$\mathbf{P16} \quad \Diamond\nabla\varphi \leftrightarrow \nabla\Diamond\varphi.$$

Proof. By **P10**, $\neg\nabla\varphi \leftrightarrow \Delta\neg\varphi$. From **MN** and **MK** we may obtain $\Box\neg\nabla\varphi \leftrightarrow \Box\Delta\neg\varphi$, and so $\Diamond\nabla\varphi \leftrightarrow \neg\Box\Delta\neg\varphi$. Instantiating **P15** with $\neg\varphi$ yields $\Box\Delta\neg\varphi \leftrightarrow \Delta\Box\neg\varphi$, and so $\neg\Box\Delta\neg\varphi \leftrightarrow \neg\Delta\Box\neg\varphi$. By instantiating **P9** with $\Box\neg\varphi$, we have $\neg\Delta\Box\neg\varphi \leftrightarrow \nabla\neg\Box\neg\varphi$, and so $\neg\Delta\Box\neg\varphi \leftrightarrow \nabla\Diamond\varphi$. Thus $\Diamond\nabla\varphi \leftrightarrow \nabla\Diamond\varphi$ by the transitivity of biconditionals. $\square$

P17 $\Diamond\nabla\varphi \rightarrow \Diamond\varphi$.

Proof. Follows immediately from **P11** and **P16**. $\square$

P18 $\Box\Delta\varphi \leftrightarrow \Box\varphi$.

Proof. For the forward direction, since $\Delta\varphi \rightarrow \varphi$ by definition and propositional logic, **MN** and **MK** yield $\Box\Delta\varphi \rightarrow \Box\varphi$. Conversely, $\Box\varphi \rightarrow \Delta\varphi$ by **P1**, and so **MN** and **MK** yield $\Box\Box\varphi \rightarrow \Box\Delta\varphi$. Since $\Box\varphi \rightarrow \Box\Box\varphi$ by **MT**, **M5**, and **MK**, we obtain $\Box\varphi \rightarrow \Box\Delta\varphi$ as desired. $\square$

P19 $\Diamond\Delta\varphi \rightarrow \Diamond\varphi$.

Proof. Since $\Delta\varphi \rightarrow \varphi$ by propositional logic, we get $\neg\varphi \rightarrow \neg\Delta\varphi$ by contraposition. By **MN** and **MK**, we obtain $\Box\neg\varphi \rightarrow \Box\neg\Delta\varphi$, and so $\Diamond\Delta\varphi \rightarrow \Diamond\varphi$ by contraposition. $\square$

P20 $\Diamond\varphi \leftrightarrow \Diamond\nabla\varphi$.

Proof. Instantiating **P18** with $\neg\varphi$ yields $\Box\Delta\neg\varphi \leftrightarrow \Box\neg\varphi$. By **P10**, $\neg\nabla\varphi \leftrightarrow \Delta\neg\varphi$, and so from **MN** and **MK** we obtain $\Box\neg\nabla\varphi \leftrightarrow \Box\Delta\neg\varphi$. Therefore $\Box\neg\nabla\varphi \leftrightarrow \Box\neg\varphi$ by transitivity of biconditionals, and so $\Diamond\nabla\varphi \leftrightarrow \Diamond\varphi$ by propositional logic. $\square$

P21 $\nabla\Diamond\varphi \rightarrow \Diamond\nabla\varphi$.

Proof. Follows immediately from **P11** and **P20**. $\square$

P22 $\Diamond\nabla\varphi \rightarrow \nabla\Diamond\varphi$.

Proof. By **P5**, we have $\Diamond\nabla\varphi \rightarrow \Delta\Diamond\varphi$. Since $\Delta\Diamond\varphi \rightarrow \Diamond\varphi$ by conjunction elimination, and $\Diamond\varphi \rightarrow \nabla\Diamond\varphi$ by definition, we obtain $\Diamond\nabla\varphi \rightarrow \nabla\Diamond\varphi$ as desired. $\square$

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