The Construction of Possible Worlds

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Abstract

This paper presents the sound and complete *Logic for Tense and Modality* (**TM**) for reasoning about future contingency in non-deterministic dynamical systems. Taking *world states* to be total configuration states of a given system, the *task relation* encodes the possible transitions between world states over a *duration*. The sentences of a bimodal language are evaluated at a *possible world* and *time* where possible worlds are functions from durations to world states that satisfy the task relation. By contrast with two-dimensional semantic theories which take both times and worlds to be primitive, the perpetuity principle that what is sometimes the case is possible is a theorem in **TM**. The paper reviews the history of tense and modal logic in §**2** to motivate the construction of possible worlds in §**3** and resulting theory of future contingency in §**4**. The *Appendix* provides formal results in §**5** which are formally verified in the Lean 4 repository for this paper. Section §**1** begins by characterizing the interaction of tense and modality.

Keywords: Dynamical Systems, Tense, Modality, Bimodal Logic, Task Semantics

1 Introduction

Non-deterministic dynamical systems leave the future open to contingency. Letting *world states* represent the total configuration states of a given system, we may model non-deterministic systems with a *task relation* that encodes the possible transitions between world states over a *duration*. This paper defines *possible worlds* as the paths through the space of world states, each of which is a complete history for the system in question. Whereas modal operators quantify over possible worlds, tense operators quantify over the past and future times in a possible world. The *Logic of Tense and Modality* (**TM**) presented below completely axiomatizes logical consequence for a language with both tense and modal operators, providing resources for reasoning about future contingency in non-deterministic dynamical systems.

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Insofar as there are *temporary sentences* which are true at some times and false at others in the same possible world, the truth-value of a temporary sentence is not fixed by a possible world alone. For instance, suppose that the sentence ‘Cary is reading’ was true this morning but false in the afternoon. Merely specifying a world of evaluation does not determine a truth-value for the sentence ‘Cary is reading’. Taking possible worlds to be temporally extended requires the sentences of a bimodal language to be evaluated relative to both a possible world and a duration.

One response denies that there are temporary sentences, assuming instead that every sentence, perhaps implicitly, includes a reference to some time or other. An *eternalist* of this kind takes the sentence ‘Cary is reading’ to be incomplete and so to be replaced by the *permanent sentence* ‘Cary is reading at t’ where t is a time.[1] However, including singular terms which refer to *times* makes an ontology of times a part of the topic of conversation. Although some sentences may be about certain times either imagined, fictional, experienced, theorized about, or otherwise discussed, the sentence ‘Cary is reading’ does not concern any time whatsoever, but rather has Cary and her engagement reading as the entire subject-matter. Moreover, since a permanent sentence that is true in τ is true at any time in τ, a permanent sentence φ is equivalent to the result of embedding φ under arbitrary tense operators, obviating the need to include tense operators in the language.[2] However, the truth-condition for the sentence ‘Cary is reading’ differs substantially from the result of embedding this sentence under tense operators. For instance, it might be true that Cary is reading, and false that she always has been reading, or is always going to be reading. Instead of excluding tense operators and temporary sentences such as ‘Cary is reading’ from the language, this paper provides a truth-conditional semantics which respects the natural morphology of tensed claims, describing their interaction with modality without positing unspoken references to an ontology of times.

Rather than including singular terms for times in the logical form of the sentences of a language, Montague [1] and Kaplan [2] evaluate sentences at world-time pairs, taking both possible worlds and times to be primitive. Assigning sentences to sets of world-time pairs makes times endogenous to the interpretation of the language.[3] This strategy provides a two-dimensional semantics for $\boxed{\scriptstyle P}$ and $\boxed{\scriptstyle F}$ which read ‘It has always been the case that’ and ‘It is always going to be the case that’, respectively. By evaluating sentences at world-time pairs, it is easy to distinguish the truth-condition for ‘Cary is reading’ from the result of embedding this sentence under tense operators which shift the time of evaluation while holding the world of evaluation fixed. Whereas ‘Cary is reading’ is assigned to the set of all world-time pairs in which Cary is reading in that world at that time, ‘Cary always was reading’ is assigned to the set of all world-time pairs where she is reading at all earlier times in that world.

Although two-dimensional semantics admits distinct truth-conditions for sentences of the form φ, $\boxed{\scriptstyle P}\varphi$, $\boxed{\scriptstyle F}\varphi$, $\Diamond_{P}\varphi$, $\Diamond_{F}\varphi$ and sentences with iterated temporal operators, taking possible worlds to be structureless points within a model while nevertheless

[1] Atomic permanent sentences could be regimented by ‘$R_t(c)$’ or ‘$R(c,t)$’. What matters is that the time t is included in the logical form rather than as a parameter at which the sentence is evaluated.

[2] There are pathological cases if time is bounded, though these exceptions are beside the present point.

[3] Alternatively, sentences could be assigned to characteristic functions from world-time pairs to truth-values, or else functions from times (perhaps together with other contextual parameters) to functions from worlds to truth-values, all of which take times to be endogenous to the interpretation of the language.

representing temporally extended histories comes at both an intuitive and theoretical cost. Letting $\langle\mathrm{P}\rangle\varphi \coloneqq \neg\boxed{\mathrm{P}}\neg\varphi$ and $\langle\mathrm{F}\rangle\varphi \coloneqq \neg\boxed{\mathrm{F}}\neg\varphi$ express ⌜It has been the case that φ⌝ and ⌜It is going to be the case that φ⌝, I will take $\triangle\varphi \coloneqq \boxed{\mathrm{P}}\varphi \wedge \varphi \wedge \boxed{\mathrm{F}}\varphi$ and $\triangledown\varphi \coloneqq \langle\mathrm{P}\rangle\varphi \vee \varphi \vee \langle\mathrm{F}\rangle\varphi$ to read ⌜It is always the case that φ⌝ and ⌜It is sometimes the case that φ⌝ where $\Box$ and $\Diamond$ are the metaphysical modals which shift the world of evaluation while holding the time fixed. A semantics of this kind makes the following *perpetuity principles* invalid:

P1 $\Box\varphi \to \triangle\varphi$.

P2 $\triangledown\varphi \to \Diamond\varphi$.

So long as sentence letters may be assigned to any set of world-time pairs where there is more than one time, φ may be true in every world σ at a time x without also being true in a world τ at every time x'. Insofar as $\Box\varphi$ is true in a world τ at time x just in case φ is true in σ at time x for every world σ, it follows that **P1** is invalid, and so **P2** is also invalid on account of being logically equivalent. However, it is natural to assume that whatever is metaphysically necessary is always the case, or equivalently, whatever is sometimes the case is metaphysically possible. Rather than taking both possible worlds and times to be primitive, this paper constructs possible worlds so that the perpetuity principles are not invalidated.

To bring out the metaphysical reading of the modal operators $\Box$ and $\Diamond$ that I will assume, consider Alvin who recently turned ten years old: $\langle\mathrm{P}\rangle\texttt{Ten}$. Although there is a time where Alvin turned ten— i.e., $\triangledown\texttt{Ten}$— one might deny that it is *now* possible for Alvin to turn ten— i.e., $\neg\Diamond\texttt{Ten}$— and so **P2** is false. The invalidity of **P1** and **P2** may then be taken to be an advantage of two-dimensional semantics by accommodating such counterexamples. This example highlights that $\Box$ and $\Diamond$ cannot be assimilated to the natural language expressions 'necessarily' and 'possibly' which quantify over a contextually restricted set of possible worlds rather than all objective possible worlds. For instance, it is often natural to restrict consideration to some relevant subset of possible worlds that overlap with the past leading up to the present. By contrast, the metaphysical modals quantify over all possible worlds, including those that bear no relation to the actual past from our present perspective. In the case of Alvin turning ten years old, we may consider possible worlds just like the world imagined to be actual, in which Alvin turned ten years old some time ago but where events are shifted forward such that he is just now turning ten years old. Insofar as the metaphysical modal operators quantify over all such possible worlds, it follows immediately that it *is* possible for Alvin to turn ten years old, though not in any possible world that shares our actual past leading up to and including the present.

Taking metaphysical modality to be the strongest objective modality, **§5.1** provides its formal characterization in the *Appendix*.[4] To clarify what objectivity requires, it will help to contrast other objective operators such as nomological necessity with epistemic operators such as 'Homer believes that', which are not objective. Letting ⌜$\varphi \equiv \psi$⌝ read ⌜For it to be the case that φ just is for it to be the case that ψ⌝ where ⌜$\vec{\chi}_{(\psi/\varphi)}$⌝ is the result of freely substituting ψ for φ among the arguments $\vec{\chi}$, an n-place operator $\mathcal{Q}$ is *objective* only if all instances of the following schema hold:

[4] See Williamson [3–5] for discussion. The theory of the objective modalities in **§5.1** establishes existence in **T1** and uniqueness up to equivalence in **L4**, comparing Bacon's [6] broadest necessity in **R3**.

Transparency: $(\varphi \equiv \psi) \rightarrow [\mathcal{Q}(\vec{\chi}) \equiv \mathcal{Q}(\vec{\chi}_{(\psi/\varphi)})]$.[5]

Letting $\boxed{\scriptstyle N}$ read 'It is nomologically necessary that', I will take $\boxed{\scriptstyle N}$ to be an objective modal operator that is strictly weaker than the metaphysical modals. Rather than quantifying over all possible worlds, $\boxed{\scriptstyle N}\varphi$ is true in a world τ at time x just in case φ is true in σ at x for every world σ which conforms to the laws of nature in τ at x. Whereas $\Box\varphi \rightarrow \boxed{\scriptstyle N}\varphi$ is valid, $\boxed{\scriptstyle N}\varphi \rightarrow \Box\varphi$ admits counterexamples insofar as there are metaphysically possible worlds that do not obey the laws of nature. In contrast to the metaphysical and nomological modals, 'Homer believes that' is not transparent, and so is not an objective operator. For instance, 'Homer believes that Hesperus is rising' and 'Homer believes that Phosphorus is rising' may differ in truth-value despite the fact that Hesperus is Phosphorus and so 'Hesperus is rising' and 'Phosphorus is rising' express the same proposition, though this was unknown to Homer.

By taking world states, durations, and tasks to be primitive in **§3**, this paper presents an intensional *task semantics* for a bimodal language $\mathcal{L}$ suited to reasoning about future contingency.[6] Sentences are evaluated at a possible world and duration, where possible worlds are defined to be functions from durations to world states that satisfy the task relation. By contrast with two-dimensional theories that take both possible worlds and times to be primitive, the task semantics validates the perpetuity principles **P1** and **P2**, vindicating the thesis that whatever is sometimes the case is possible. Not only are the perpetuity principles theorems of **TM**, but the discrete, dense, and Dedekind-complete extensions of **TM** are completely axiomatized, where the soundness and completeness results are formally verified in the Lean 4 repository for this paper.[7] After presenting **TM** and its extensions, **§4** provides an account of the openness of the future before drawing connections to dynamical systems theory and adjacent logics in computer science. The *Appendix* in **§5** collects the formal results referred to throughout. Although **§2** motivates the task semantics and resulting logic by reviewing the twentieth-century history of tense and modal reasoning, readers may skip to the main construction in **§3**, which stands on its own.

2 Primitive Worlds

In order to provide a flexible semantics that characterizes the modal systems that Lewis and Langford [8] first set out in Appendix II of their 1932 textbook, Kripke [9, 10] took the models of a modal language to include a nonempty set of *possible worlds* W, an *evaluation world* w, an *accessibility relation* R, and an *interpretation* $\mathcal{I}$ assigning each sentence letter to a truth-value at each world. Adding these resources improved on Carnap's [11, 12] semantic theories that evaluated sentences at *state-descriptions* which, for any atomic sentence of the language, include that sentence or its negation but not both. Carnap [11] specified how state-descriptions are to be understood:

[5] Transparency is a necessary condition of objectivity. In **§5.1** Leibniz's Law is restricted so that transparency is postulated of the objective modalities in **O-Tran** in **D5** rather than derived.

[6] I provide a hyperintensional task semantics in Brast-McKie [7] in which world states are defined.

[7] The Lean 4 repository is available at https://github.com/benbrastmckie/BimodalLogic.

> A state description is a class of sentences which represents a possible specific state of affairs by giving a complete description of the universe of individuals with respect to all properties and relations designated by predicates in the system.[8] (p. 50)

Since Carnap's state-descriptions are entirely syntactic in their construction and so determined by the primitive symbols included in a non-modal language, Carnap does not provide a genuine model theory but rather a single structure by which to interpret a modal language. Building on Carnap's efforts, Kripke [14] first sought to evaluate sentences at *complete assignments* over a *domain* D where each assignment maps the singular terms to elements in D and n-adic predicates to sets of n-tuples of elements in D where sentence letters are assigned to truth or falsity. By contrast with Carnap's state-descriptions which belong to a single fully specified structure, Kripke took the domain D included in a model to be any set whatsoever, where it was by quantifying over all such models that Kripke defined validity for the language. Given that there are no two complete assignments that agree on all elements of the language, the range of complete assignments is determined by the primitive symbols included in the language together with the domain provided by the models of the language.

Despite the language relativity of both Carnap's state-descriptions and Kripke's complete assignments modeled after them, these constructions sought to represent *possibilities* to provide semantic clauses for modal operators by quantifying over all or some possibilities. Whereas objective modality concerns possible ways for a system to be configured independent of the expressive resources included in any language, *interpretational modality* concerns consistent ways of interpreting a language, making a degree of language-relativity both inherent and appropriate. Although the language relativity of Carnap [11, 12] and Kripke's [14] early accounts presents no issue for the interpretational modals that they may be taken to have described, Kripke sought to accommodate a broader range of readings for the modal operators by decoupling the possibilities over which the modal operators quantified from the primitive symbols in the language.[9] Beginning with the propositional fragment, Kripke [9] introduced a primitive set of *possible worlds* W which he took to be any nonempty set whatsoever, where the interpretation of the language was handled by an *interpretation function* mapping each sentence letter and possible world to a truth-value.[10] By also specifying a primitive relation R over W for *relative possibility*, Kripke showed how to associate the reflexivity, symmetry, and transitivity constraints on R with the corresponding **T**, **B**, and **4** axioms which characterized the differences between the most prominent modal systems that Lewis and Langford [8] had introduced.[11] Although Kripke [10]

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[8] In his earlier work, Carnap [13, p. 95] writes, "If a false sentence is not L-false, hence not self-contradictory, it describes a situation which [is] possible though not real," and later that, "a system S has to do with many objects, and hence we have to consider the possible states of affairs of all the objects dealt with in S and with respect to all properties, relations, etc., dealt with in S," (p. 101) indicating Carnap's concern with an objective modality that quantifies over *possible circumstances* rather than the interpretational modality that the formal details of Carnap's theory are better equipped to support.

[9] Kripke [15, 16] went on to describe a metaphysical reading of the modal operators for which it is inappropriate to relativize the range of possibilities to the expressive power of a language.

[10] Officially, Kripke [9] took the *models* on a *model structure* $\langle W, R, w\rangle$ to be functions which take a propositional variable and world to a truth-value. I have replaced propositional variables with sentence letters and reserved the label 'model' to improve consistency with the definitions presented below.

[11] The relational structures underlying Kripke semantics were anticipated algebraically by Jónsson and Tarski's representation theorem for Boolean algebras with operators [17, 18], which embeds every such algebra into a powerset algebra over a set with a binary relation— i.e., a frame. However, neither Tarski

went on to extend his semantics to include predicates and first-order quantifiers, it will suffice for present purposes to restrict consideration to the propositional fragment of the languages with which Carnap and Kripke were concerned.

Despite their differences, neither Carnap nor Kripke were focused on interpreting languages with tense operators or singular terms for an ontology of times.[12] Insofar as state-descriptions, complete assignments, and possible worlds are taken to determine the truth-values of temporary sentences, it is implausible to interpret these elements as temporally extended histories. Besides being irrelevant to the interpretation of a modal language without tense operators, taking possible worlds to be temporally extended underdetermines the truth-values for temporary sentences, requiring the addition of a temporal parameter to the point of evaluation in order to specify the time in the possible world at which the sentence is to be evaluated. However, neither Carnap nor Kripke provide any indication that times are to be considered in evaluating sentences. Since the possible worlds that Kripke took to be primitive cannot be interpreted as temporally extended, they must be instantaneous.[13]

Although it is natural to conceive of possible worlds as complete configurations of a system at a moment when interpreting the sentences of a language with either tense operators or modal operators but not both, the same cannot be said for bimodal languages with both tense and modal operators. Not only do instantaneous moments fail to specify what comes before or after when considered on their own, including an ordering of moments once and for all fails to capture the range of different orderings of moments that are possible. What is needed to interpret bimodal sentences is an encoding of both the modal and temporal dimensions of the semantics.

While working to develop a semantics for tense logic, Prior [20, 22] took the first steps towards constructing the resources needed to interpret a bimodal language. Rather than restricting consideration to operators that quantify over instantaneous configurations of a system, Prior defined histories to consist of temporally ordered configuration states. In addition to the tense operators which quantify over the past and future states in a history, Prior introduced operators that quantify over histories to characterize the open future. The following section will review Prior's efforts, highlighting his most important insight that histories are to be defined rather than primitive while also considering the limitations this approach faces.

2.1 World States

In *Time and Modality*, Prior [20] developed a Diodorean interpretation of $\Diamond\varphi$ at a sequence of numbers which he took to represent the future.[14] Whereas the first number

nor his contemporaries recognized the connection to modal semantics. Tarski reportedly told Kripke in 1962 that he could not see a link between their respective programs. See Goldblatt [19] for historical discussion.

[12]Although Kripke interprets possible worlds as *moments* in a letter discussing Prior's [20] temporal interpretation of the modal operators, he writes, "I myself was working with ordinary modal logic." For the published correspondence and accompanying discussion, see Ploug [21, p. 373].

[13]In the only example that Kripke [10] provides, the sentence 'Sherlock Holmes is bald' has a truth-value even though a time has not been specified and the name 'Sherlock Holmes' does not refer. Assuming Cary to be in the flesh and blood, 'Cary is reading' may be said to have a truth-value with considerably less controversy. In both cases, Kripke's semantics returns a truth-value despite the fact that a time has not been included in the sentence in question nor among the parameters at which that sentence is interpreted.

[14]Diodorus Cronus (died c. 284 BCE) was a Megarian logician who defined possibility and necessity in temporal terms, where a proposition is possible just in case it either is or will be true, and necessary just in case it is true and always will be true. Although Prior was concerned to provide a semantics and logic for

in the sequence represented the truth-value of φ in the present, the subsequent numbers represent the truth-value of φ at incrementally later times.

Commenting on Prior's book, Kripke observed in a letter [21, p. 370] that the Diodorean system validates $\boxed{\mathrm{F}}\Diamond_{\mathrm{F}}\varphi \vee \boxed{\mathrm{F}}\Diamond_{\mathrm{F}}\neg\varphi$ which does not belong to an S4 logic. Kripke went on to suggest a branching structure in which the past is determined but the future remains open, where models of this kind are more appropriate to the S4 logic that Prior had discussed. Crediting Kripke for this account in [22, p. 27], Prior developed a number of semantic theories and corresponding logics for tensed languages where sentences are evaluated at *world states* which model, "instantaneous total states of the world," (p. 88) rather than temporally extended histories. Intuitively, each world state is a complete configuration of the system under study at an instant, individuated by the intrinsic properties of that system in question. In order to interpret the tense operators, Prior included a strict *earlier-than* relation $<$ to order the world states. Situating his semantic approach in contrast to previous developments, Prior [22, p. 7-8] cites Broad's [23, p. 315] criticism of McTaggart's [24] famous argument that time is unreal, writing that the fundamental error in McTaggart's argument is his attempt to, "impose conditions appropriate to a tenseless language upon a tensed one." Prior [22, p. 12] avoids this defect by following Broad's suggestion to, "drop the temporal predicates 'past', 'present', and 'future'."[15] Whereas McTaggart includes times, temporal predicates, and tense operators all in one language in order to present his arguments, Prior interprets an object language with tense operators by appealing to an extensional metalanguage in which world states are ordered by an earlier-than relation but no other temporal relations or properties are included.

Although the world states form a strict total order in one of the simplest versions of his semantics, Prior goes on to consider models which accommodate an open future by taking the world states to form a strict partial order that is connected and left-linear rather than total.[16] Despite these differences, $<$ is a strict partial order on each account that Prior considered since no world state is permitted to be earlier than itself, and a world state s is earlier than r if both s is earlier than t and t is earlier than r. It is important to observe that taking world states to be strictly ordered prevents the same world state from occurring more than once in any history for that system. However, supposing that history never repeats itself is a substantive assumption which should not be built into the semantics. Moreover, there are systems which we may wish to study that admit loops in their evolution. For instance, taking the system in question to be a chessboard, there are histories for that system which include meandering end games where the same board state occurs more than once. Nothing should rule out consideration of such games of chess. Rather, this example highlights a fundamental

a tensed language without metaphysical modal operators, he stops short of claiming that the metaphysical modals are definable in terms of the temporal operators with which he was concerned.

[15] Instead of following Reichenbach [25] in distinguishing between the point-of-speech, point-of-reference, and point-of-event to evaluate tensed sentences at a single world state, it is by evaluating sentences with nested tense operators that the evaluation time shifts between the right number of world states.

[16] A *strict partial order* $\langle T, < \rangle$ is a set T equipped with an irreflexive and transitive relation $<$. Letting $x \sim y \coloneqq (x < y) \vee (x = y) \vee (x > y)$ express that x and y are comparable, a strict partial order $\langle T, < \rangle$ is *total* if $s \sim t$ for any $s, t \in T$. A strict partial order $\langle T, < \rangle$ is *left-linear* just in case $s \sim t$ for any $s, t, r \in T$ where $s < r$ and $t < r$. Letting $\overset{*}{\sim}$ be the transitive closure of $\sim$, a frame $\langle T, < \rangle$ is *connected* just in case $x \overset{*}{\sim} y$ for all $x, y \in T$. Writing $z \leqslant x$ for $z < x \vee z = x$, a left-linear frame is connected just in case for any $x, y \in T$ there is some z where $z \leqslant x$ and $z \leqslant y$.

limitation of the theoretical roles that world states play in Prior's semantics. Although it is natural to consider systems which occupy the same instantaneous configuration at different times, this is forbidden if world states are strictly ordered.

To disentangle the distinct theoretical roles which world states play in Prior's semantics, it will help to define a minimally constrained class of models that generalize on Thomason's [26] reconstruction of the Peircean and Ockhamist semantic theories that Prior [22] developed. Letting $\mathcal{L}^{\mathrm{T}} \coloneqq \langle \mathbb{L}, \bot, \to, \boxed{\scriptstyle\mathrm{P}}, \boxed{\scriptstyle\mathrm{F}} \rangle$ be a non-modal propositional language, a *strict model* of $\mathcal{L}^{\mathrm{T}}$ is an ordered triple $\mathcal{P} = \langle T, <, |\cdot| \rangle$ where $\langle T, < \rangle$ is a strict partial order for a nonempty set of world states T, and $|p_i| \subseteq T$ for all sentence letters $p_i \in \mathbb{L}$.[17] Although some strict models correspond to a single history for a system by also being total, strict models may accommodate distinct histories which do not intersect or only partially overlap. A *strict history* in a strict model $\langle T, <, |\cdot| \rangle$ is any maximal total suborder $h_i = \langle T_i, <_i \rangle$ of $\langle T, < \rangle$.[18] Letting $H_{\mathcal{P}}$ be the set of all strict histories of a strict model $\mathcal{P}$, we may take $H^x_{\mathcal{P}} \coloneqq \{\langle T_i, <_i \rangle \in H_{\mathcal{P}} \mid x \in T_i\}$ to be the strict histories $\langle T_i, <_i \rangle$ of $\mathcal{P}$ which include the world state $x \in T_i$.

Despite being defined rather than primitive, strict histories play a similar role to possible worlds in the semantics that Montague [1] and Kaplan [2] went on to provide since each strict history corresponds to a complete temporal evolution of the system in question. In particular, consider the following semantic clauses:

Peircean: $\mathcal{P}, x \vDash \boxed{\scriptstyle\mathrm{P}}^{\mathrm{P}}\varphi$ *iff* $\mathcal{P}, y \vDash \varphi$ for some $h_i \in H^x_{\mathcal{P}}$ and all $y \in h_i$ where $y <_i x$.
$\mathcal{P}, x \vDash \boxed{\scriptstyle\mathrm{F}}^{\mathrm{P}}\varphi$ *iff* $\mathcal{P}, y \vDash \varphi$ for some $h_i \in H^x_{\mathcal{P}}$ and all $y \in h_i$ where $x <_i y$.

Ockhamist: $\mathcal{P}, h_i, x \vDash \boxed{\scriptstyle\mathrm{P}}^{\mathrm{O}}\varphi$ *iff* $\mathcal{P}, h_i, y \vDash \varphi$ for all $y \in T_i$ where $y <_i x$.
$\mathcal{P}, h_i, x \vDash \boxed{\scriptstyle\mathrm{F}}^{\mathrm{O}}\varphi$ *iff* $\mathcal{P}, h_i, y \vDash \varphi$ for all $y \in T_i$ where $x <_i y$.

Whereas Prior [22, p. 126, 132] provided a semantics for the metric tense operators and Thomason [26] restricted attention to $\Diamond_{\mathrm{P}}^{\mathrm{P}}$ and $\Diamond_{\mathrm{F}}^{\mathrm{P}}$, I have derived the semantics for $\boxed{\scriptstyle\mathrm{P}}^{\mathrm{P}}$ and $\boxed{\scriptstyle\mathrm{F}}^{\mathrm{P}}$ from the semantics that Thomason provides while generalizing the account to accommodate all strict models of $\mathcal{L}^{\mathrm{T}}$.[19] The Peircean and Ockhamist semantics disagree about what the tense operators express at any world state that belongs to more than one strict history. Whereas the Peircean semantics quantifies over all past or future world states in *some* strict history that includes the world state at which the tensed sentence is evaluated, the Ockhamist semantics evaluates tensed sentences at both a strict history and world state in that history, quantifying over just the world states in that strict history. As a result, the Peircean semantics has a number of unnatural consequences. For instance, given a board state in a game of chess where there is at least one history in which Black does not make any further blunders, the sentence 'Black is not going to blunder' is true even though there may be other histories in which Black goes on to blunder. This is far from natural. Moreover, revising the Peircean semantics to quantify over *all* strict histories in addition to all past or future

[17] I will assume $\neg\varphi \coloneqq \varphi \to \bot$, $\varphi \vee \psi \coloneqq \neg\varphi \to \psi$, $\varphi \wedge \psi \coloneqq \neg(\varphi \to \neg\psi)$, and $\varphi \leftrightarrow \psi \coloneqq (\varphi \to \psi) \wedge (\psi \to \varphi)$.

[18] Letting a *total suborder* of $\langle T, < \rangle$ be any total order $\langle T_i, <_i \rangle$ where $T_i \subseteq T$ and $<_i$ is the restriction of $<$ to T_i, a total suborder $\langle T_i, <_i \rangle$ of $\langle T, < \rangle$ is *maximal* just in case $\langle T_j, <_j \rangle = \langle T_i, <_i \rangle$ for any total suborder $\langle T_j, <_j \rangle$ of $\langle T, < \rangle$ where $\langle T_i, <_i \rangle$ is a total suborder of $\langle T_j, <_j \rangle$.

[19] The Peircean semantics given above can be simplified were one to restrict consideration to strict models which are left-linear (similarly right-linear), since world states would then have a unique past (future).

world states makes the operators $\boxed{\scriptstyle\mathrm{P}}^{\mathrm{P}}$ and $\boxed{\scriptstyle\mathrm{F}}^{\mathrm{P}}$ too strong and their duals too weak. For instance, in a game of chess in which there is at least one future in which the white king is in checkmate and at least one future in which the black king is in checkmate, both 'Black is going to win' and 'White is going to win' come out true.

Rather than quantifying over all past or future world states in either some or all strict histories, the Ockhamist semantics evaluates sentences at both a strict history and world state. The sentences 'Black is going to win' and 'White is going to win' cannot both be true since there is no way for both kings to be in checkmate in the same game. In addition to claiming this advantage, another prominent difference between the Ockhamist and Peircean semantics is the expressive power that they each afford. Since the Ockhamist tense operators only quantify over the world states in the strict history included in the point of evaluation, the Peircean operators may be defined in terms of the Ockhamist operators given the following *stability operator*:

$(\boxdot^{\mathrm{O}})$ $\mathcal{P}, h_i, x \vDash \boxdot^{\mathrm{O}}\varphi$ *iff* $\mathcal{P}, h_j, x \vDash \varphi$ for all $h_j \in H^x_{\mathcal{P}}$.

By contrast to the Ockhamist tense operators, the stability operator quantifies over all intersecting strict histories that occupy the same world state at which the sentence is evaluated. Prior [22, p. 125] reads '$\boxdot^{\mathrm{O}}\varphi$' as 'It is now unpreventable that φ', though uses 'Necessarily φ' for convenience, claiming that whereas the Peircean's, "rather strong 'will be' is simply the Ockhamist 'necessarily will be', the Ockhamist 'will be' [is] untranslatable," (p. 130) for the Peircean.[20] As Prior observes, $\varphi \to \boxdot^{\mathrm{O}}\varphi$ is valid for non-temporal sentences, where $\boxed{\scriptstyle\mathrm{P}}$ (similarly $\boxed{\scriptstyle\mathrm{F}}$) may be included in φ by restricting to strict models that are left-linear (right-linear). Although the Ockhamist may define the Peircean tense operators $\boxed{\scriptstyle\mathrm{P}}^{\mathrm{P}}\varphi \coloneqq \Diamonddot^{\mathrm{O}}\boxed{\scriptstyle\mathrm{P}}^{\mathrm{O}}\varphi$ and $\boxed{\scriptstyle\mathrm{F}}^{\mathrm{P}}\varphi \coloneqq \Diamonddot^{\mathrm{O}}\boxed{\scriptstyle\mathrm{F}}^{\mathrm{O}}\varphi$, the Peircean is not in a position to define the Ockhamist operators for tense, nor to provide a semantics for a stability operator since sentences are evaluated at world states alone.

Despite their differences, neither the Peircean nor Ockhamist semantics permit the same world state in T to occur more than once in a strict history, nor are there strict histories in which the same world states in T occur in different orders. Rather, $\langle T, < \rangle$ is required to be a strict partial order for any strict model $\langle T, <, |\cdot| \rangle$ of $\mathcal{L}^{\mathrm{T}}$, and so if $s <_i t$ for any strict history of $\langle T, <, |\cdot| \rangle$, then $t \nless_j s$ for all strict histories of $\langle T, <, |\cdot| \rangle$. However, given Prior's conception of world states as complete configurations, there are systems in which the same world states occur more than once or in different orders in different possible histories. As brought out above, a chess game may include the same board state more than once, or two chess games may agree in all respects with the exception of a transposition of board states which occur in a different order.[21]

Insofar as T is taken to be the set of all world states of a particular system under study, the strict histories for that system do not permit the same world states to occur more than once or in different orders, significantly compromising the range of applications for the semantics. Rather than accepting these limitations, it is natural to reject Prior's conception of T as the set of world states where these are taken to be complete configurations of the system, taking T to be the set of *times* in accordance

[20] Prior [22, p. 130] attributes these observations to J.M. Shorter in 1957 but does not provide a proof.

[21] Following Müller's [27] suggestion, Rumberg [28] extends Prior [22] and Thomason's [26] semantics. By virtue of working over a partial order, this extension faces the same difficulties brought out here.

with their strict ordering. After all, T is intended to provide resources for articulating a semantics for temporal operators which quantify over what comes before or after each element in T. On this reading, the truth-conditions for the sentences of $\mathcal{L}^{\mathrm{T}}$ specify *when* a sentence is true without indicating which world state the system happens to occupy at each time. Strict models may include strict histories in which the same world states occur more than once or in different orders for the simple reason that strict models only concern the times at which sentences are evaluated without saying anything about the world states that the system occupies at those times. As a result, there is no telling which times in T occupy the same world states, or which strict histories include the same world states in different orders. For instance, supposing the white king to be in check at multiple times in a game of chess, there is no way to distinguish which times in the truth-condition for 'The white king is in check' correspond to the same complete configuration of the chessboard (if any) and which times correspond to distinct configurations. Moreover, given two strict histories for a chess game which agree in all respects with the exception of a short sequence of moves in which the same board states occur in different orders, the divergent sequences must include distinct times in T despite reordering the same configurations of the chessboard.

Taking T to be the set of times rather than world states avoids ruling out strict histories in which the same world states occur more than once or in different orders. Since all that is required to interpret tense operators is to specify *when* each sentence is true, the erroneous interpretation of T as a set of complete configurations of a system may be forgiven when considering non-modal languages like $\mathcal{L}^{\mathrm{T}}$. Despite providing an adequate range of semantic primitives for interpreting the tensed languages with which Prior and Thomason were concerned, the same cannot be said for a bimodal language with operators for both tense and metaphysical modality. In order to evaluate a sentence φ of the bimodal language $\mathcal{L}$ at a model $\mathcal{P}$, strict history h_i, and time x, it is important that $x \in T_i$ be a valid time in the strict history $h_i = \langle T_i, <_i \rangle$. As a result, the strongest modal operator that an Ockhamist model theory can provide only quantifies over those strict histories in which the time of evaluation occurs, and so cannot quantify over the full range of strict histories included in a strict model. This is exactly what the Ockhamist stability operator $\boxdot^{\mathrm{O}}$ achieves, though Prior is careful not to confuse this modality with metaphysical necessity since the stability modal does not quantify over all strict histories whatsoever. Since there is no Ockhamist modality that is able to quantify over all strict histories, the range of strict histories cannot represent the full range of possible evolutions for the system in question.

The source of these limitations is that Prior takes the elements in T to answer two different questions: *when* a sentence is true which is given by a position in a strict order, and *what* configuration the system then occupies. Since the position-marker and the configuration are one and the same, no configuration can occupy two positions, which is why the same world state cannot recur in any strict history. Rather than accepting these limitations, I will take both *world states* and *durations* to be semantic primitives in **§3**, separating the theoretical roles that Prior conflates. By also including a *task relation*, I will define possible worlds as appropriately constrained functions from durations to world states, tracing out different paths through the space of all world states in alignment with standard definitions in dynamical systems theory. In addition to providing a natural model theory in which possible worlds may occupy the same world states more than once or in different orders, taking sets of world states to provide truth-conditions for the sentence letters in the language makes times exogenous to the interpretation of the language. As I show in **C5** of the *Appendix*, both **P1** and **P2** are valid over a natural class of models that I define for the bimodal language.

Instead of constructing the space of possible worlds from world states, tasks, and times, Montague [1] and Kaplan [2] separate the *when* from the *what* by taking both the set of possible worlds W and the set of times T to be primitive along with a weak total order $\leqslant$ for the *at least as early as* relation.[22] Abstracting from their differences, I will take a *two-dimensional model* $\mathcal{M}_2 = \langle W, T, \leqslant, |\cdot|\rangle$ to include an interpretation function where $|p_i| \subseteq W \times T$ is a set of world-time pairs for each sentence letter p_i with $i \in \mathbb{N}$. However natural it may seem to extend Kripke's semantics along these lines, Montague's semantics undermines the expressive power of the modal operator included in his language while Kaplan's semantics invalidates the perpetuity principles, weakening the logic that the resulting model theory supports. The following subsections will review the shortcomings of these early bimodal semantic theories before turning to present the construction of possible worlds in **§3**.

2.2 Necessarily Always

Whereas Montague [1] sought to provide a formal semantics for a fragment of English by regimenting that fragment in a quantified bimodal language for which he gave a recursive semantics, I will restrict consideration to the propositional fragment of Montague's formal language. To facilitate comparison, I will include $\boxtimes$ in place of $\Box$ in the propositional language $\mathcal{L}^{\text{M}} = \langle \mathbb{L}, \bot, \to, \boxtimes, \boxed{\text{P}}, \boxed{\text{F}}\rangle$. By defining the well-formed sentences of $\mathcal{L}^{\text{M}}$ in the usual way, we may extend the interpretation provided by each two-dimensional model $\mathcal{M}_2$ to all well-formed sentences of $\mathcal{L}^{\text{M}}$ as follows:

(p_i) $\mathcal{M}_2, w, x \vDash p_i$ *iff* $\langle w, x\rangle \in |p_i|$.

($\bot$) $\mathcal{M}_2, w, x \nvDash \bot$.

($\to$) $\mathcal{M}_2, w, x \vDash \varphi \to \psi$ *iff* $\mathcal{M}_2, w, x \nvDash \varphi$ or $\mathcal{M}_2, w, x \vDash \psi$.

($\boxtimes$) $\mathcal{M}_2, w, x \vDash \boxtimes\varphi$ *iff* $\mathcal{M}_2, u, y \vDash \varphi$ for all $u \in W$ and $y \in T$.

($\boxed{\text{P}}$) $\mathcal{M}_2, w, x \vDash \boxed{\text{P}}\varphi$ *iff* $\mathcal{M}_2, w, y \vDash \varphi$ for all $y \in T$ where $y < x$.

($\boxed{\text{F}}$) $\mathcal{M}_2, w, x \vDash \boxed{\text{F}}\varphi$ *iff* $\mathcal{M}_2, w, y \vDash \varphi$ for all $y \in T$ where $x < y$.

Postponing the controversies that beset the interpretation of the future tense operator to **§4.1**, I will take the semantics for both the extensional and tense operators to be uncontroversial for the time being. Whereas Montague [1] took modal claims of the form ⌜$\boxtimes\varphi$⌝ to be read ⌜It is necessarily the case that φ⌝, the same paper in the edited collection included an editor's note by Richard Thomason [29, p. 259, FN 9] that $\boxtimes$, "is interpreted in the sense of 'necessarily always'." After all, $\boxtimes\varphi$ is true in a world w

[22] Little turns on whether the order is weak $\leqslant$ or strict $<$ following Prior. I will follow Montague and Kaplan in omitting consideration of the accessibility relation between possible worlds.

at a time x just in case φ is true in all worlds at all times, quantifying over both the modal and temporal dimensions of Montague's semantics at once.

By letting a *moment* be any ordered pair $\langle w, x\rangle \in W \times T$, the semantics for $\boxtimes$ quantifies over all moments. By contrast, I will take the following *universal semantics* for $\boxed{2}$ to quantify over all worlds while leaving the temporal parameter unchanged:

($\boxed{2}$) $\mathcal{M}, w, x \vDash \boxed{2}\varphi$ *iff* $\mathcal{M}, u, x \vDash \varphi$ for all $u \in W$.

To evaluate whether $\boxed{2}$ or $\boxtimes$ expresses metaphysical necessity, I will take $\mathcal{L}^{\mathrm{K}}$ to be the result of replacing $\boxtimes$ in $\mathcal{L}^{\mathrm{M}}$ with $\boxed{2}$ while maintaining the other operators included in the language. Next we may derive the semantic clause for $\triangle$ from its definition above:

($\triangle$) $\mathcal{M}, w, x \vDash \triangle\varphi$ *iff* $\mathcal{M}, w, y \vDash \varphi$ for all $y \in T$.

Extending $\mathcal{L}^{\mathrm{K}}$ to include $\boxtimes$ as a primitive symbol makes $\boxtimes\varphi \leftrightarrow \boxed{2}\triangle\varphi$ valid by being true in every world at every time on any two-dimensional model, thereby rendering $\boxtimes$ redundant. Rather, I will maintain $\boxtimes\varphi \coloneqq \boxed{2}\triangle\varphi$ as a metalinguistic abbreviation in $\mathcal{L}^{\mathrm{K}}$, deriving the semantics for $\boxtimes$ from its definition. Since **T3** in the *Appendix* proves that $\boxed{2}$ cannot be defined in $\mathcal{L}^{\mathrm{M}}$, it follows that $\mathcal{L}^{\mathrm{K}}$ is more expressive than $\mathcal{L}^{\mathrm{M}}$. For the purposes of comparing $\boxed{2}$ and $\boxtimes$ in a common language, $\mathcal{L}^{\mathrm{K}}$ enjoys a clear advantage over working in $\mathcal{L}^{\mathrm{M}}$. Assuming that $\mathcal{L}^{\mathrm{M}}$ is to be replaced by $\mathcal{L}^{\mathrm{K}}$ where $\boxtimes$ is then defined in terms of $\boxed{2}$ and $\triangle$ as above, the question remains whether or not to read $\ulcorner\boxed{2}\varphi\urcorner$ as $\ulcorner$It is necessarily the case that $\varphi\urcorner$ in accordance with Thomason's suggested reading of $\ulcorner\boxtimes\varphi\urcorner$ as $\ulcorner$It is necessarily always the case that $\varphi\urcorner$, or to follow Montague.[23]

In opposition to Thomason's reading, one might appeal to the invalidity of **P1** and **P2** over the class of two-dimensional models in defense of taking $\boxtimes$ and $\Diamond_{\times}\varphi \coloneqq \neg\boxtimes\neg\varphi$ to be the metaphysical modals rather than $\boxed{2}$ and $\Diamond_{2}\varphi \coloneqq \neg\boxed{2}\neg\varphi$.[24] So long as there are multiple times in T, we may consider a model $\mathcal{M}_2$ where p_i is assigned to a set of world-time pairs that includes $\langle w', x\rangle$ for all $w' \in W$ at a fixed time $x \in T$ but that does not include $\langle w, x'\rangle$ for all times $x' \in T$ at a fixed world $w \in W$. It follows that $\boxed{2}p_i$ may be true in w at x while $\triangle p_i$ is false, thereby invalidating **P1** where **P2** is equivalent. By contrast, the following *trivial perpetuity principles* which result from replacing $\boxed{2}$ and $\Diamond_{2}$ in **P1** and **P2** with $\boxtimes$ and $\Diamond_{\times}$ may be shown to be valid:

TP $\boxtimes\varphi \to \triangle\varphi$. **TC** $\triangledown\varphi \to \Diamond_{\times}\varphi$.

Given the definition $\boxtimes\varphi \coloneqq \boxed{2}\triangle\varphi$, **TP** is an instance of the **T** axiom $\boxed{2}\psi \to \psi$ where **TC** is equivalent. Instead of expressing substantive interaction principles for tense and modality, the trivial perpetuity principles are valid over the two-dimensional models of $\mathcal{L}^{\mathrm{K}}$ for entirely modal reasons. To avoid appealing to the status of the operators in one language or another, we may put the point purely semantically:

[23] Admittedly, it is not clear that Montague [1] was concerned with metaphysical modality, though the semantics he provides for $\boxtimes$ is completely unrestricted in quantifying over times and worlds.

[24] Dorr and Goodman [30, p. 636] present considerations along these lines, though they, "think it best to avoid 'world'-talk altogether in theorizing about temporary matters" (p. 646). In opposition to this perspective, I will assume that the semantic primitives included in the models of a language ought to provide the intuitive bedrock by which to define meaningful truth-conditions for the sentences of the language.

Triviality: $(\forall w \in W \forall x \in T : \mathcal{M}, w, x \vDash \varphi) \rightarrow (u \in W \rightarrow \forall x \in T : \mathcal{M}, u, x \vDash \varphi)$.
$(u \in W \wedge \exists x \in T : \mathcal{M}, u, x \vDash \varphi) \rightarrow (\exists w \in W \exists x \in T : \mathcal{M}, w, x \vDash \varphi)$.

The principles above are instances of the first-order theorems $\forall w\Psi \rightarrow \Psi[u/w]$ and $\Psi[u/w] \rightarrow \exists w\Psi$.[25] Despite including quantification over times, nothing in Ψ accounts for why the *Triviality* principles are valid. Rather, these principles are valid on account of universal instantiation and existential generalization with respect to quantification over the worlds in W independent of the quantification over times in Ψ. That **TP** and **TC** are valid over all two-dimensional models is no compensation for failing to articulate substantive interaction principles. By contrast, the universal semantics for $\boxed{2}$ and $\diamondsuit$ enjoys a clear advantage over the Montagovian semantics for $\boxtimes$ and $\diamondsuit$ by drawing on their greater expressive power to make **P1** and **P2** substantive.

Dorr and Goodman [30, pp. 635, 655] define immediate necessity in terms of their primitive necessity operator with the resources that Fine [31] provides. By including a countable set of *time variables* $\mathcal{V} \coloneqq \{t_i : i \in \mathbb{N}\}$ in the language which may be bound by first-order quantifiers and taking an *assignment* to be any function $g : \mathcal{V} \rightarrow T$ from time variables in $\mathcal{V}$ to times in T, Dorr and Goodman define $\Box$ as follows:

($\exists t$) $\mathcal{M}, w, x, g \vDash \exists t\varphi$ *iff* $\mathcal{M}, w, x, g' \vDash \varphi$ for some g' differing from g at most in t.

($\texttt{Pr}$) $\mathcal{M}, w, x, g \vDash \texttt{Present}(t)$ *iff* $g(t) = x$.[26]

Converse Definition: $\boxed{2}\varphi \coloneqq \exists t[\texttt{Present}(t) \wedge \boxtimes(\texttt{Present}(t) \rightarrow \varphi)]$.[27]

By taking $\mathcal{L}^{\text{F}}$ to extend $\mathcal{L}^{\text{M}}$ to include the Finean resources given above, **L6** in the *Appendix* shows that $\boxed{2}\varphi$ is true in world w at time x on assignment g just in case φ is true in world u at time x on assignment g for all worlds $u \in W$. Given this alignment with the universal semantic clause provided above for $\boxed{2}$, **T4** shows that the perpetuity principles are invalid over the two-dimensional models for $\mathcal{L}^{\text{F}}$.

Whereas $\boxtimes$ is easy to define in terms of $\boxed{2}$ and $\triangle$ in $\mathcal{L}^{\text{K}}$, the ideological complexity of the *Converse Definition* reflects the relative obscurity of $\boxed{2}$ from the perspective of the primitive expressions included in $\mathcal{L}^{\text{F}}$. Given the naturalness of the semantics for $\boxed{2}$, the complexity of the *Converse Definition* indicates how awkward it is to make do with the operator $\boxtimes$ whose semantic clause quantifies over moments in $W \times T$ instead of worlds in W. Although the extra ideology adds expressive power to $\mathcal{L}^{\text{F}}$, including temporal predicates and quantifiers over times abandons Prior's insight that much of the trouble that McTaggart [24] faced was due to cobbling together tense operators, temporal predicates, and first-order quantification over times all into one language. Following Prior, I will exclude temporal predicates and first-order quantification over times from the object language, limiting use of these resources to the metalanguage which I will use to articulate truth-conditions for the sentences of a bimodal language.

[25] Replace Ψ with either: (1) $w \in W \rightarrow \forall x \in T : \mathcal{M}, w, x \vDash \varphi$; or (2) $w \in W \wedge \exists x \in T : \mathcal{M}, w, x \vDash \varphi$.

[26] Dorr and Goodman [30, p. 637] admit that Fine's [31] use of propositional quantifiers to eliminate time variables from the expanded language behaves pathologically when $\boxed{2}$ is replaced by $\boxtimes$.

[27] Dorr and Goodman [30, p. 656] refer to $\boxed{2}$ here as *immediate necessity* and are clear that they nowhere endorse model-theoretic characterizations of metaphysical necessity, Montagovian or otherwise.

In addition to avoiding the threat of making the temporal operators superfluous, this approach does not carry the same commitment to an ontology of times.[28]

Instead of validating the perpetuity principles trivially by taking the semantics for the metaphysical modals to quantify over all moments, the following subsection considers attempts to validate the perpetuity principles by constraining the class of two-dimensional models. As I will argue, the resulting theory is committed to an absolute theory of time that cannot be avoided without undermining the significance of the truth-conditions for the sentences of the language. This will set the stage for **§3** which provides the task semantics, proving that **P1** and **P2** are valid without imposing any *ad hoc* constraints on the space of models for the language.

2.3 Absolute Time

Instead of taking the semantics for the metaphysical modals to quantify over moments in $W \times T$, Kaplan's [2] semantics for $\boxdot$ quantifies over just the possible worlds W in a two-dimensional model of $\mathcal{L}^{\text{K}}$. Although Kaplan does not articulate a logic for his two-dimensional semantics, **P1** and **P2** are invalid over the two-dimensional models that Kaplan defines. Since **T5** shows that no frame constraint on $\langle W, T, \leqslant \rangle$ can validate the perpetuity principles without collapsing the temporal order to include at most one time, it remains to validate the perpetuity principles over the two-dimensional semantics by imposing a model constraint. In particular, **T6** shows that **P1** and **P2** are valid over the abundant two-dimensional models of $\mathcal{L}^{\text{K}}$ which I will define as follows:

> *Time-Shift:* The worlds $w, w' \in W$ are *time-shifted from* x *to* y— i.e., $w \approx^y_x w'$— in a model $\mathcal{M}_2$ of $\mathcal{L}^{\text{K}}$ *iff* there is an order automorphism $\bar{a} : T \to T$ where $y = \bar{a}(x)$ and the following holds for all sentence letters $p_i \in \mathbb{L}$ and times $z \in T$:
>
> $$\langle w', \bar{a}(z) \rangle \in |p_i| \;\Leftrightarrow\; \langle w, z \rangle \in |p_i|.$$ [29]
>
> *Abundance:* A two-dimensional model $\mathcal{M}_2$ of $\mathcal{L}^{\text{K}}$ is *abundant iff* for every $w \in W$ and $x, y \in T$, there is some $w' \in W$ that is time-shifted from x to y, i.e., $w \approx^y_x w'$.

Abundant models include all time-shifted worlds.[30] Paradigm examples of abundant models identify T with either the set of integers $\mathbb{Z}$, rational numbers $\mathbb{Q}$, or real numbers $\mathbb{R}$. As **T7** shows, time is unbounded in abundant models with at least two distinct times.[31] Nevertheless, it is natural to take some systems such as a game of chess to have a bounded set of times. That abundant models cannot accommodate bounded temporal orders is a significant limitation which the task semantics avoids.

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[28] Following Prior [22], one could introduce propositional quantifiers and define times as the propositions that are sometimes true and, whenever true, decide every proposition, though I will not do so here.

[29] An *order automorphism* on the structure $\langle T, \leqslant \rangle$ is a monotonic bijection $\bar{a} : T \to T$ where $\bar{a}(x) \leqslant \bar{a}(y)$ whenever $x \leqslant y$, shifting all times in T without changing their order.

[30] Not all models are abundant: letting $W = \{w\}$ and $T = \{0, 1\}$ where $|p_1| = \{\langle w, 0 \rangle\}$, there is no automorphism on T mapping 0 to 1 that preserves truth-values, and so the model is not abundant.

[31] A weak partial order $\langle T, \leqslant \rangle$ is *bounded* if both of the following hold and *unbounded* if neither holds:

Bounded Below: There is some $y \in T$ where $y \leqslant x$ for all $x \in T$.

Bounded Above: There is some $y \in T$ where $x \leqslant y$ for all $x \in T$.

Restricting to the abundant models validates the perpetuity principles without providing a corresponding frame constraint. By **T5** this is unavoidable: no condition on $\langle W, T, \leqslant \rangle$ validates **P1** and **P2** short of collapsing the temporal order. In giving up the correspondence between the perpetuity axioms and a condition on the frames, we give up the central explanatory virtue of relational semantics— that a logic's theorems are underwritten by the structural properties defining a corresponding class of frames.[32] In addition to this formal deficit, abundant models are required to include a vast set of primitive worlds. We may state this precisely with the following definition:

Temporary: A sentence φ of $\mathcal{L}^{\mathrm{K}}$ is *temporary* in a two-dimensional model $\mathcal{M}_2$ *iff* $\mathcal{M}_2, w, x \vDash \varphi$ and $\mathcal{M}_2, w, y \nvDash \varphi$ for some world w and times x and y.

Abundant models with temporary sentences include all merely temporal differences between worlds. For instance, given an abundant model $\mathcal{M}_2$ in which φ is temporary, there is a world w and times $x, y \in T$ where $\mathcal{M}_2, w, x \vDash \varphi$ and $\mathcal{M}_2, w, y \nvDash \varphi$, and so by *Abundance* there is a world w' that is time-shifted $w \approx_x^y w'$ from x to y. As shown in **L8**, what is true in w at x is exactly the same as what is true in w' at y. Since $w \approx_x^y w'$, it follows more generally that there is some order automorphism $\bar{a} : T \to T$ where $w \approx_z^{\bar{a}(z)} w'$ for all $z \in T$ by **L7**. Again by **L8**, it follows that for every time $z \in T$ exactly the same sentences that are true in w at z are true in w' at $\bar{a}(z)$. Insofar as there are intended models where W represents a meaningful range of possible histories of events without ever representing the same history more than once, it follows that abundant models with temporary sentences include all merely temporal differences between otherwise indistinguishable histories of events.

I will take *temporal absolutism* to be the claim that there are merely temporal differences between otherwise indistinguishable histories— $w \approx_x^y w'$ for some $w \neq w'$— a thesis that I will assume it is natural to resist. In addition to the formal deficit indicated above, requiring the models of $\mathcal{L}^{\mathrm{K}}$ to be abundant gives rise to an unfortunate trade-off: either one must embrace temporal absolutism or else admit an excess of possible worlds that represent the same possible history many times over. Although it is preferable to avoid redundancies among the semantic primitives included in a model, one might attempt to defend a restriction to the abundant models by taking the time-shifted worlds to be empty artifacts. However, insofar as sets of world-time pairs are to provide meaningful truth-conditions for the well-formed sentences of $\mathcal{L}^{\mathrm{K}}$, the worlds in W cannot be entirely void of significance. For instance, an *abundance theorist* might attempt to maintain intelligible truth-conditions by abstracting from the merely temporal differences by way of the following definitions:

Time-Shifted Worlds: $w \approx w'$ *iff* $w \approx_x^y w'$ for some $x, y \in T$.

World Abstraction: $[w] \coloneqq \{w' \in W \mid w \approx w'\}$.

Possible Worlds: $W_{\approx} \coloneqq \{[w] \subseteq W \mid w \in W\}$.

[32]That the perpetuity principles are not frame-definable over two-dimensional frames is an instance of the general problem that interaction axioms in products of modal logics pose for frame definability (see Gabbay, Kurucz, Wolter, and Zakharyaschev [32, p. 221]). By contrast, compare the correspondence theorems for the tense operators given in **§3.3** following van Benthem [33].

By letting $[w]$ be the set of time-shifted worlds that represent the same possible world as w, an abundance theorist might claim that $W_\approx$ represents the range of genuinely distinct possible worlds rather than the primitive set W. Although an abundant model which admits temporary sentences is guaranteed to include merely temporal differences between the primitive worlds in W, the elements in $W_\approx$ abstract from the merely temporal differences between worlds. An abundance theorist may take **T6** to show how to validate **P1** and **P2** by restricting consideration to the abundant models without embracing absolutism or undermining the interpretation of possible worlds.

Despite providing a method for identifying a reduced range of genuinely distinct possible worlds for each abundant model of $\mathcal{L}^{\text{K}}$, it is important to observe that $W_\approx$ is defined in terms of the interpretation of the sentence letters of the language. Even if the worlds $w \approx w'$ are time-shifted in a model $\mathcal{M}_2$ of $\mathcal{L}^{\text{K}}$ and so $[w] = [w']$, it does not follow that $w \approx w'$ in any other model $\mathcal{M}_2'$ of $\mathcal{L}^{\text{K}}$. For instance, assuming $w \approx_x^y w'$ in $\mathcal{M}_2$ for just $x, y \in T$ where $x \neq y$ and $\langle w, x\rangle \in |p_0|$, there is an automorphism $\bar{a} : T \to T$ where $y = \bar{a}(x)$ and $\langle w', y\rangle \in |p_0|$. By letting $\mathcal{M}_2'$ be identical to $\mathcal{M}_2$ except for taking $\langle w', y\rangle \notin |p_0|$, it follows that $w \not\approx_x^y w'$ in $\mathcal{M}_2'$ for any $x, y \in T$, and so $[w] \neq [w']$ in $\mathcal{M}_2'$. This construction demonstrates how the equivalence classes in $W_\approx$ may expand or contract depending on the truth-conditions assigned to the sentence letters by a model $\mathcal{M}_2$ of the language $\mathcal{L}^{\text{K}}$. The same construction shows that *Abundance* itself may be destroyed by changing the interpretation of a single sentence letter, confirming that it is a condition on models rather than frames since it is not preserved under a change of valuation. As a result, an abundance theorist cannot appeal to $W_\approx$ in order to specify meaningful truth-conditions for the sentence letters of $\mathcal{L}^{\text{K}}$ without circularity. In particular, consider the following definitions:

Time-Shifted Moments: $[w, x] \coloneqq \{\langle u, y\rangle \mid w \approx_x^y u\}$.

Truth-Conditional Abstraction: $|p_i|_\approx \coloneqq \{[w, x] \mid \langle w, x\rangle \in |p_i|\}$.

The first definition identifies the genuinely distinct moments in $\mathcal{M}_2$ by abstracting from the differences between moments which make the same sentence letters true in $\mathcal{M}_2$. However, for $|p_i|_\approx$ to provide a meaningful truth-condition for p_i in $\mathcal{M}_2$, one must already have an independent grasp of the original truth-condition $|p_i|$ provided by $\mathcal{M}_2$, undermining the need to specify $|p_i|_\approx$ in the first place. In addition to circularity, taking the truth-conditions for the sentence letters of $\mathcal{L}^{\text{K}}$ to be the basis upon which to identify the range of genuinely distinct possible worlds $W_\approx$ puts the cart before the horse. Rather, the *intended models* of the language aim to provide a range of meaningful semantic primitives by which to specify truth-conditions for the language, not the other way around. Whereas an *instrumentalist* gives up the ambition to admit intended models, I will take it to be an advantage of any semantic theory to avoid undermining the interpretation of the semantic primitives for that theory.

It is worth comparing a similar strategy applied to the semantics of a first-order extensional language $\mathcal{L}^1$. Given a domain D that represents a meaningful range of primitive *objects*, a truth-conditional semantics may interpret $\mathcal{L}^1$ by specifying the extensions of the constants and n-place predicates as elements of D and subsets of D^n. For instance, we may interpret a constant c in a model by assigning it to an element of D and interpret a one-place predicate F in a model by taking its extension to be a subset of the domain D. However, if it is denied that D includes genuinely distinct objects but rather some other entities which may represent the same genuinely distinct object many times over, we lose our initial grasp on the meaning of the constant c and the predicate F. If D is left uninterpreted, or else interpreted in terms of another assignment of constants and predicates to extensions in D^n, then we cannot look to D to provide an independent basis upon which to interpret the language $\mathcal{L}^1$.

In keeping with a standard methodology in truth-conditional semantics, I will take the semantic primitives in an intended model for a given object language to provide an independently meaningful basis upon which to make sense of the truth-conditions for that language. Rather than presuming that the only way for the semantic primitives in an intended model to be meaningful is to be identical to the parts of the reality that they model, I will take the intended models of a language to provide an idealization that simulates what that object language is able to express with the resources of a well-theorized metalanguage. Whereas the object language is of primary metaphysical significance, the metalanguage provides expressive resources for simulating what the object language is able to express rather than a description of the entities to which the object language is implicitly committed. I will refer to this approach as *simulation metasemantics* in opposition to *realist metasemantics* which takes the intended model to include the constituents of reality that the object language describes as well as to *instrumentalist metasemantics* in which the semantic primitives have no significance beyond the instrumental role that they play in the semantics.[33]

To validate the perpetuity principles without undermining the significance of the truth-conditions for the language, the following section presents an alternative to two-dimensional semantics that is compatible with a simulation metasemantics. Instead of taking possible worlds to be primitive points devoid of any internal structure, I will provide a *task semantics* that constructs possible worlds from *world states*, *tasks*, and *durations*. As I show, the resulting models provide a natural first-order simulation of what the bimodal language is able to express while also validating **P1** and **P2**.

3 Possible Worlds

To interpret the bimodal language $\mathcal{L} \coloneqq \langle \mathbb{L}, \bot, \rightarrow, \Box, \lhd, \rhd \rangle$, well-formed sentences will be evaluated at a *possible world* and a *duration* in addition to a model of $\mathcal{L}$. Whereas the languages $\mathcal{L}^{\mathrm{T}}$, $\mathcal{L}^{\mathrm{M}}$, $\mathcal{L}^{\mathrm{K}}$, and $\mathcal{L}^{\mathrm{F}}$ considered above take the tense operators $\boxed{\scriptstyle P}$ and $\boxed{\scriptstyle F}$ to be primitive, $\mathcal{L}$ takes the *since* $\lhd$ and *until* $\rhd$ operators that Kamp [35] introduced to be primitive, defining $\boxed{\scriptstyle P}$ and $\boxed{\scriptstyle F}$ below. Rather than following Montague [1] and Kaplan [2] in taking possible worlds to be primitive, I will define possible worlds to be functions from durations to world states constrained by the task relation.[34] Whereas

[33] Cresswell [34] shows that tense operators supplemented with selective indexical devices— operators that store and retrieve evaluation indices— are expressively equivalent to first-order quantification over times, extending the result to modal operators and possible worlds. Cresswell concludes that "the [ontological] commitment is genuine if the range of sentences in which these constructions occur requires a semantics equivalent in power to explicit variable binding" (p. 2). The bimodal language $\mathcal{L}^{\mathrm{K}}$ includes no such selective devices, and its operators are all unselective in Cresswell's sense. However, even if $\mathcal{L}^{\mathrm{K}}$ were enriched with such devices, I do not follow Cresswell in taking semantics to be any guide to ontological commitment.

[34] Although the world states will be primitive for our purposes, I define world states in Brast-McKie [7], appealing to both the task and *parthood* relations to provide a semantics for counterfactual conditionals.

world states are the instantaneous maximal possible configurations of the system under study, the durations parameterize the possible trajectories through the space of world states. Thus I will take the durations and not the world states to form a total order while positing additive group structure so that durations can be added and subtracted. Each duration x is used to indicate the world state a possible world occupies after x duration from the origin, where this justifies the convention of referring to durations as *times* relative to a possible world, though often left implicit. The *interpretation function* included in a model of $\mathcal{L}$ maps each sentence letter in $\mathcal{L}$ to a set of world states, providing the truth-condition for that sentence by specifying the configurations of the system in which that sentence is true. Since each set of world states represents the ways for the system to be without including any temporal elements, durations are strictly exogenous to the interpretation of the sentence letters in $\mathcal{L}$, where this fact plays a critical role in validating the perpetuity principles.

Positing a set of *world states* W in addition to a set of *durations* D distinguishes what state a system is in from when the system is in a given state. A *temporal order* is defined over D to be any nontrivial totally ordered abelian group $\mathcal{D} = \langle D, +, 0, \leqslant\rangle$.[35] Instead of admitting the incomparable times permitted by a partial order, $\mathcal{D}$ is used to define the space of possible evolutions for a system where time is totally ordered in each. Since not every path through the space of world states is possible, frames include a *parameterized task relation* $w \Rightarrow_x u$ to encode which transitions between world states $w, u \in W$ over a positive duration $x \geqslant 0$ are possible. Defining reverse transitions by the *reflection convention* $w \Rightarrow_{-x} u \coloneqq u \Rightarrow_x w$ for $x \geqslant 0$, the task relation determines the following for any world states $w, v \in W$ and durations $x, y \in D$:

Fiber: $\texttt{fib}(w, x) \coloneqq \{u \in W : w \Rightarrow_x u\}$.

Cone: $(w)_x \coloneqq \bigcup_{|y|<x} \texttt{fib}(w, y)$ where $x > 0$.

Segment: $[w, v]_x^y \coloneqq \texttt{fib}(w, x) \cap \texttt{fib}(v, -y)$ where $x, y \geqslant 0$.

The task relation encodes what is dynamically possible for the system in question. Each fiber collects the world states accessible from some w over a duration x, where negative durations flip the direction of accessibility. Whereas a cone collects the world states accessible from w within x in either temporal direction by taking the union of all fibers for w over $(-x, x)$, a segment collects all world states u accessible from both w of x duration in the past and v of y duration in the future.[36]

[35] A *group* is any $\langle G, \cdot, 1_G\rangle$ where: (1) $a \cdot b \in G$ whenever $a, b \in G$; (2) $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ for all $a, b, c \in G$; (3) $1_G \in G$ where $a \cdot 1_G = 1_G \cdot a = a$ for all $a \in G$; and (4) for each $a \in G$, there is some $-a \in G$ where $a \cdot (-a) = 1_G$, written $a - a = 1_G$ for ease. A group is *nontrivial* just in case $a \neq 1_G$ for some $a \in G$. A group is *abelian* just in case $a \cdot b = b \cdot a$ for all $a, b \in G$. A group is *totally ordered* by $\leqslant$ just in case $\leqslant$ is a total order where for all $a, b, c \in G$, if $a \leqslant b$, then $a \cdot c \leqslant b \cdot c$.

[36] Rather than merely recording which transitions are possible, a *stochastic task function* assigns to each pair of world states $w, u \in W$ and duration $x \geqslant 0$ a transition probability $\Phi_x(w, u) \in [0, 1]$ where $\sum_{u \in \texttt{fib}(w,x)} \Phi_x(w, u) = 1$ whenever the fiber $\texttt{fib}(w, x)$ is nonempty and countable. A stochastic analogue of the *Compositionality* constraint below would then require $\Phi_{x+y}(w, v) = \sum_{u \in [w,v]_x^y} \Phi_x(w, u)\, \Phi_y(u, v)$ for all world states $w, v \in W$ and durations $x, y \geqslant 0$, composing the transition probabilities through the intermediate world states in the segment $[w, v]_x^y$ in the manner of the Chapman–Kolmogorov equation. Since uncountable fibers require the distributions to give way to probability measures over the fibers and the sums to integrals, stating either constraint in full generality calls for measurable structure on W that the present framework does not otherwise require, and so I will not develop this extension here.

Letting a nonempty family of sets $\mathcal{S}$ be $\supseteq$-*directed* just in case $S \subseteq S_1 \cap S_2$ for some $S \in \mathcal{S}$ whenever $S_1, S_2 \in \mathcal{S}$, I will define a *task frame* to be any $\mathcal{F} = \langle W, \mathcal{D}, \Rightarrow \rangle$ where W is a nonempty set of world states, $\mathcal{D}$ is a temporal order, and $\Rightarrow$ is a task relation satisfying the following for positive durations $x, y \geqslant 0$:

Compositionality: $w \Rightarrow_{x+y} v$ if and only if $w \Rightarrow_x u$ and $u \Rightarrow_y v$ for some $u \in W$.

Seriality: $w \Rightarrow_x u$ and $v \Rightarrow_x w$ for some $u, v \in W$.

Limit: $\bigcap_{x>0} (w)_x = \{w\}$.

Saturation: $\bigcap \mathcal{S} \neq \varnothing$ for any $\supseteq$-directed family $\mathcal{S}$ of nonempty fibers and segments.

The frame constraints above require tasks to compose through intermediate world states, where no world state is a dead end in either direction, distinct world states are instantaneously separated, and no world state is missing wherever compatible constraints converge.[37] For any nonempty set of durations $X \subseteq D$, a *partial history* is a function $\tau : X \to W$ where $\tau(x) \Rightarrow_{y-x} \tau(y)$ for all $x, y \in X$. Consider extending a partial history τ by assigning a world state $u \in W$ to a further duration $z \notin X$. Each assignment $\tau(x) = w$ for $x \in X$ requires $u \in \texttt{fib}(w, z-x)$ for any candidate u. As **L11** shows in the *Appendix*, fibers imposed from the same side of z nest by *Compositionality*, and so nearer assignments subsume the constraints imposed from farther away. When a nearest past assignment $\tau(x) = w$ and a nearest future assignment $\tau(y) = v$ flank z, the candidates are confined to the segment $[w, v]^{y-z}_{z-x}$ where segments nest toward z. Nevertheless, X may contain no assignment nearest to z on a given side— as when z is a limit of times in X— and so the nested constraints imposed on z may not have a least member. However, *Seriality* and *Compositionality* ensure that every fiber and segment imposed on z is nonempty as shown in **L12**, where **L13** shows that they form a $\supseteq$-directed family of nonempty sets. *Saturation* then secures a common candidate, where **L16** shows that partial histories may be extended to new durations.

Task frames are a special case of labeled transition systems (LTS) in which the transitions are labeled by durations, providing what is also called *non-deterministic dynamical systems*.[38] Setting these connections aside until **§4.2**, a *convex history* is any partial history $\tau : X \to W$ whose domain X is convex, thereby settling which world state occurs at each time throughout a fixed interval.[39] For instance, drawing on $\mathbb{Z}$ to provide durations, an individual game of chess may be identified with a convex history beginning with the initial board state and ending after a finite number of moves. A *world history* is any convex history $\tau : X \to W$ whose domain is *total*, so that

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[37]The set of *basic opens* $B_{\mathcal{F}} := \{(w)_x : w \in W, x \in D, \text{ and } x > 0\}$ generates a topology $\mathcal{T}_{\mathcal{F}} := \langle W, \mathcal{O}_{\mathcal{F}} \rangle$ where the set of *open sets* $\mathcal{O}_{\mathcal{F}}$ is the result of closing $B_{\mathcal{F}}$ under arbitrary union and finite intersection. The *Appendix* proves in **T8** that $\mathcal{T}_{\mathcal{F}}$ is *T1*, and hence *R0* as shown in **C2**.

[38]A labeled transition system is a triple $\langle S, \Lambda, \to \rangle$ consisting of a set S of states, a set Λ of labels, and a transition relation $s \xrightarrow{a} s'$ where $s, s' \in S$ and $a \in \Lambda$. The relation need not be functional, since a state may have several a-labeled successors or none, which is exactly what allows the task relation $\Rightarrow_x$ to carry indeterminism. See Baier and Katoen [36] for a comprehensive treatment.

[39]A subset $X \subseteq D$ is *convex* just in case $y \in X$ whenever $x, z \in X$ and $x < y < z$. A partial history is *difference closed* just in case the differences $x - y$ between times $x, y \in X$ form a subgroup of D to preserve the duration structure of the domain while permitting internal gaps.

$X = D$.[40] The *Appendix* proves in **T9** that every partial history extends to a world history, and so the bounded convex histories may be identified with restrictions of world histories. By extending the one-point partial history given by the zero-duration loop $w \Rightarrow_0 w$ for any world state $w \in W$, **C3** shows that every world state occurs at some time in some world history, thereby excluding idle world states. Since partial histories, convex histories, and world histories differ only in their domains, I will refer to these generically as *histories* wherever the distinction is immaterial.

In contrast with the strictly ordered world states of **§2.1**, nothing prevents a world state from occurring at many times in a single history so that $\tau(x) = \tau(y)$ for $x \neq y$. Although the durations in D form a total order so that a history passes through each time in its domain at most once, no such constraint governs the world states that a history assigns to each time. In the case of a meandering end game in chess, there is nothing to prevent the chessboard from returning to a board state that it has already occupied. Similarly, nothing prevents two histories τ and σ from passing through the same world states in a different order between two times $x < y$, where $\tau(z) = \sigma(z)$ for all $z \notin (x, y)$, as in chess games which transpose move order. Because a world state may occur more than once in a history, and world states may occur in different orders in different histories, there may be no single answer to the question of which world states come before or after another. Rather, it is by specifying a time x in a history τ that we may determine which world states occur before or after x in τ.[41]

Instead of positing an abundance of primitive time-shifted possible worlds, the same semantic primitives included in a task frame $\mathcal{F}$ generate an abundance of histories. Writing $H_{\mathcal{F}}$ for the set of all world histories defined over the task frame $\mathcal{F}$, **C3** proves $H_{\mathcal{F}}$ is nonempty. Since $\mathcal{D}$ is translation invariant, we may define the following for any world histories $\tau, \sigma \in H_{\mathcal{F}}$ and durations $x, y \in D$:

Time-Shift: $\tau \approx^y_x \sigma$ *iff* $\tau(z) = \sigma(z + y - x)$ for all $z \in D$.

Possible World: $[\tau]_{\mathcal{F}} \coloneqq \{\sigma \in H_{\mathcal{F}} \mid \tau \approx^y_x \sigma$ for some $x, y \in D\}$.

By taking the equivalence class $[\tau]_{\mathcal{F}}$ of world histories time-shifted from τ to represent a *possible world*, one might define $\mathbb{W}_{\mathcal{F}} \coloneqq \{[\tau]_{\mathcal{F}} \mid \tau \in H_{\mathcal{F}}\}$ to be the set of all possible worlds for the task frame $\mathcal{F}$. Whereas $\mathbb{W}_{\mathcal{F}}$ encodes the distinct paths through the space of all world states W, the world histories parameterize those paths by assigning times to world states while acknowledging that the choice of origin is arbitrary. Although one might take $\mathbb{W}_{\mathcal{F}}$ to hold a metaphysical standing that $H_{\mathcal{F}}$ cannot, it is $H_{\mathcal{F}}$ that will play an important role in the semantics. Since the classes in $\mathbb{W}_{\mathcal{F}}$ will play no further role below, I will also refer to $H_{\mathcal{F}}$ as the set of *possible worlds*.

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[40] Every nontrivial ordered group is infinite and unbounded: given $d > 0$, translation invariance yields the strictly ascending chain $0 < d < 2d < 3d < \cdots$, and inverses give $\cdots < -3d < -2d < -d < 0$.

[41] A growing block view in the spirit of Broad [37] might replace history-time pairs with *pasts* defined as ordered sequences of world states leading up to the present. Although no past recurs, a past may contain the same world state more than once, and transposed chess games are distinct pasts that end in the same world state. Since a past fixes the course of events leading up to the present, the separate parameters for tense and modal reasoning collapse, leaving future-directed operators to quantify over the extensions of a past from the present much as in the Peircean semantics of **§2.1**. A view of this kind encodes an asymmetry between a settled past and an open future, though the parameterization is lost: whereas a history and time determine a unique past, no past determines the times at which its world states occur.

Despite being defined rather than primitive, possible worlds and times play the same roles in the semantics that they have traditionally played in bimodal frameworks. A *model* of $\mathcal{L}$ is any tuple $\mathcal{M} = \langle W, \mathcal{D}, \Rightarrow, |\cdot|\rangle$ where $\mathcal{F} = \langle W, \mathcal{D}, \Rightarrow\rangle$ is a task frame and $|p_i| \subseteq W$ for every sentence letter $p_i \in \mathbb{L}$. The well-formed sentences of $\mathcal{L}$ are evaluated at a possible world $\tau \in H_{\mathcal{F}}$ and time $x \in D$ included in a model $\mathcal{M}$ by way of the following semantic clauses:

(p_i) $\mathcal{M}, \tau, x \vDash p_i$ *iff* $\tau(x) \in |p_i|$.

($\bot$) $\mathcal{M}, \tau, x \nvDash \bot$.

($\rightarrow$) $\mathcal{M}, \tau, x \vDash \varphi \rightarrow \psi$ *iff* $\mathcal{M}, \tau, x \nvDash \varphi$ or $\mathcal{M}, \tau, x \vDash \psi$.

($\Box$) $\mathcal{M}, \tau, x \vDash \Box\varphi$ *iff* $\mathcal{M}, \sigma, x \vDash \varphi$ for all $\sigma \in H_{\mathcal{F}}$.

($\lhd$) $\mathcal{M}, \tau, x \vDash \varphi \lhd \psi$ *iff* $\mathcal{M}, \tau, z \vDash \psi$ for some time $z < x$ where $\mathcal{M}, \tau, y \vDash \varphi$ for all intermediate times $y \in D$ where $z < y < x$.

($\rhd$) $\mathcal{M}, \tau, x \vDash \varphi \rhd \psi$ *iff* $\mathcal{M}, \tau, z \vDash \psi$ for some time $z > x$ where $\mathcal{M}, \tau, y \vDash \varphi$ for all intermediate times $y \in D$ where $x < y < z$.

Whereas the semantics for the metaphysical modal $\Box$ quantifies over possible worlds in $H_{\mathcal{F}}$, the semantics for $\lhd$ and $\rhd$ quantifies over times in D. Letting $\top \coloneqq \bot \rightarrow \bot$, we may define the tense operators $\Diamond_{\!P}\varphi \coloneqq \top \lhd \varphi$ and $\Diamond_{\!F}\varphi \coloneqq \top \rhd \varphi$ in $\mathcal{L}$, where $\boxed{\scriptstyle P}\varphi \coloneqq \neg\Diamond_{\!P}\neg\varphi$ and $\boxed{\scriptstyle F}\varphi \coloneqq \neg\Diamond_{\!F}\neg\varphi$. Since $\top$ holds at every time, the guard in the clauses for $\lhd$ and $\rhd$ is trivially satisfied by $\top$. By contrast, the *next* $\textcircled{\scriptsize F}\varphi \coloneqq \bot \rhd \varphi$ and *previous* $\textcircled{\scriptsize P}\varphi \coloneqq \bot \lhd \varphi$ operators require φ to hold at an immediate successor or predecessor time since $\bot$ holds at no intermediate time. Thus $\textcircled{\scriptsize F}\varphi$ and $\textcircled{\scriptsize P}\varphi$ are equivalent to $\bot$ whenever there is no next or previous time, respectively. It is easy to derive the following clauses:

($\boxed{\scriptstyle P}$) $\mathcal{M}, \tau, x \vDash \boxed{\scriptstyle P}\varphi$ *iff* $\mathcal{M}, \tau, y \vDash \varphi$ for all $y \in D$ where $y < x$.

($\boxed{\scriptstyle F}$) $\mathcal{M}, \tau, x \vDash \boxed{\scriptstyle F}\varphi$ *iff* $\mathcal{M}, \tau, y \vDash \varphi$ for all $y \in D$ where $x < y$.

($\textcircled{\scriptsize P}$) $\mathcal{M}, \tau, x \vDash \textcircled{\scriptsize P}\varphi$ *iff* $\mathcal{M}, \tau, y \vDash \varphi$ for some $y < x$ where $y \geqslant z$ for all times $z < x$.

($\textcircled{\scriptsize F}$) $\mathcal{M}, \tau, x \vDash \textcircled{\scriptsize F}\varphi$ *iff* $\mathcal{M}, \tau, y \vDash \varphi$ for some $y > x$ where $y \leqslant z$ for all times $z > x$.

The operators $\lhd$ and $\rhd$ provide greater expressive power than $\boxed{\scriptstyle P}$ and $\boxed{\scriptstyle F}$ which cannot define $\textcircled{\scriptsize F}$ and $\textcircled{\scriptsize P}$ over $\mathbb{Z}$-time. As Kamp [35] showed, $\lhd$ and $\rhd$ are expressively complete over $\mathbb{Z}$-time and $\mathbb{R}$-time. Any condition on a time that can be stated in first-order logic with the temporal order and a unary predicate for each sentence letter is expressed over $\mathbb{Z}$-time and $\mathbb{R}$-time by some sentence of $\mathcal{L}$. Although Kamp's theorem does not extend to $\mathbb{Q}$-time, $\lhd$ and $\rhd$ presume nothing about the structure of time, in contrast to $\textcircled{\scriptsize F}$ and $\textcircled{\scriptsize P}$ which are only of interest over discrete temporal orders.[42]

[42] Citing Dana Scott, Prior [22, p. 66] introduces operators for *tomorrow* and *yesterday*, represented here by $\textcircled{\scriptsize F}$ and $\textcircled{\scriptsize P}$, though without providing a semantics. Prior [22, p. 126, 132] focuses instead on the *metric tense operators* $Pn\alpha$ and $Fn\alpha$ which assert that α holds at a distance n in the past or future. To incorporate metric tense operators, $\mathcal{L}$ would need to include singular terms for durations, which I will not explore here.

Consider a possible world α for the chessboard in which Black nearly checkmates White on move 31 before blundering the dark squared bishop. Playing on until move 47 in α, Black manages to win the end game, saying:

(K) If I hadn't blundered my bishop, I would have won much sooner.

Although the present framework is not equipped to interpret tensed counterfactual conditionals, it is clear that Black is lamenting the existence of another possible game in which the blunder had not occurred.[43] Nevertheless, we may evaluate the following:

(P) Black could have checkmated White much sooner.

Given a sufficiently strong reading of 'could', we may regiment P in $\mathcal{L}$ as $\Diamond\Diamond_{\mathrm{P}} p_w$ where p_w reads 'White is in checkmate'. The sentence $\Diamond\Diamond_{\mathrm{P}} p_w$ is true in α at move 47 just in case there is a possible world β where $\Diamond_{\mathrm{P}} p_w$ is true at move 47. Although the finite game in which the blunder is avoided may end before move 47, that game is a bounded convex history which extends to a possible world $\beta \in H_{\mathcal{F}}$ by **T9** in which there is a 47$^{\text{th}}$ time at which to evaluate $\Diamond_{\mathrm{P}} p_w$ despite occurring after the game ends. For instance, if p_w is true at move 31 in β, then $\Diamond_{\mathrm{P}} p_w$ is true in β at move 47.[44]

Having provided a theory of truth for the language $\mathcal{L}$, it remains to provide a theory of logical consequence by which to survey the valid forms of reasoning warranted by the semantics. I will take the definition to assume the following standard form:

Logical Consequence: $\Gamma \vDash \varphi$ *iff* for any model $\mathcal{M}$ of $\mathcal{L}$, possible world $\tau \in H_{\mathcal{F}}$, and time $x \in D$, if $\mathcal{M}, \tau, x \vDash \gamma$ for all $\gamma \in \Gamma$, then $\mathcal{M}, \tau, x \vDash \varphi$.

A well-formed sentence φ of $\mathcal{L}$ is *valid* just in case $\varnothing \vDash \varphi$, dropping set notation for convenience. In addition to providing a general framework for semantic theorizing, the task semantics validates a simple and strong logic for $\mathcal{L}$ without imposing additional frame constraints. As **C5** shows, the perpetuity principles are valid:

P1 $\Box\varphi \to \triangle\varphi$. **P2** $\triangledown\varphi \to \Diamond\varphi$.

Suppose for contradiction that **P1** has a counterinstance, and so for some well-formed sentence φ, it is metaphysically necessary that φ and yet it is sometimes not the case that φ. Put semantically, $\mathcal{M}, \tau, x \vDash \Box\varphi$ and $\mathcal{M}, \tau, x \nvDash \triangle\varphi$, and so $\mathcal{M}, \tau, y \nvDash \varphi$ for some $y \in D$. We may then define the possible world $\sigma(z) = \tau(z - x + y)$, where $\sigma \in H_{\mathcal{F}}$ since the possible worlds are closed under translation **L17**. Since $\tau \approx_y^x \sigma$, it follows by **L18** that $\mathcal{M}, \sigma, x \nvDash \varphi$. Thus it follows that $\mathcal{M}, \tau, x \nvDash \Box\varphi$, contradicting the above. This proves that **P1** is valid where **P2** is equivalent.

Instead of admitting countermodels to **P1** and **P2** as in Kaplan's [2] semantics, the present approach validates the perpetuity principles without imposing *ad hoc* model constraints that undermine the significance of the truth-conditions for the language.

[43] I develop a hyperintensional semantics for tensed counterfactual conditionals in Brast-McKie [7].

[44] Alternatively, one might evaluate sentences at any convex history τ together with a time $x \in \mathtt{dom}(\tau)$, taking $\Box$ to quantify over all convex histories σ where $x \in \mathtt{dom}(\sigma)$, restricting $\boxed{\mathrm{P}}$ and $\boxed{\mathrm{F}}$ to the times in $\mathtt{dom}(\tau)$, and adapting logical consequence to replace D with $\mathtt{dom}(\tau)$. Since evaluating sentences only at possible worlds excludes no course of events by **T9**, the present account retains the stronger tense logic.

Not only does the present approach maintain the standard semantic clauses for both tense and modal operators, **P1** and **P2** support an account of metaphysical modality as the strongest objective modality whereby we may deny that φ is metaphysically necessary if φ ever fails to be the case. Moreover, taking the metaphysical modals to quantify over possible worlds rather than world-time pairs provides a more expressive theory than Montague's [1] semantics. In addition to these merits, defining possible worlds in terms of the world states, tasks, and durations avoids the need to choose between instrumentalism and temporal absolutism about a primitive ontology of time-shifted worlds. I will provide further abductive support for the task semantics given above by presenting a logic for $\mathcal{L}$ in **§3.2**. The following subsection will begin by extending $\mathcal{L}$ to include a stability operator in order to demonstrate the power of the present approach in contrast to two-dimensional semantic theories.

3.1 Restricted Modalities

The semantics for metaphysical necessity $\Box$ quantifies over all possible worlds in $H_{\mathcal{F}}$ defined over the task frame $\mathcal{F}$, thereby validating an S5 modal logic.[45] Although this is in keeping with the interpretation of metaphysical modality as the strongest objective modality, there are applications which call for restricted modals. For instance, given a world $\tau \in H_{\mathcal{F}}$ and time $x \in D$, we may let $\langle\tau\rangle_x \coloneqq \{\sigma \in H_{\mathcal{F}} \mid \sigma(x) = \tau(x)\}$ be the set of possible worlds that intersect τ at x in order to provide the following semantics:

($\boxdot$) $\mathcal{M}, \tau, x \vDash \boxdot\varphi$ *iff* $\mathcal{M}, \sigma, x \vDash \varphi$ for all $\sigma \in \langle\tau\rangle_x$.

A sentence $\boxdot\varphi$ is true in a world τ at a time x just in case φ is true at time x in every world that occupies the same world state as τ at x.[46] Letting $\diamondcdot\varphi \coloneqq \neg\boxdot\neg\varphi$, the *stability operators* $\boxdot$ and $\diamondcdot$ may be used to define the following bimodal operators:

Will Always: $\boxed{!}\varphi \coloneqq \boxdot\boxed{\text{F}}\varphi$. *Could Always:* $\boxed{?}\varphi \coloneqq \diamondcdot\boxed{\text{F}}\varphi$.

Will Eventually: $\Diamond_{!}\varphi \coloneqq \boxdot\Diamond_{\text{F}}\varphi$. *Could Eventually:* $\Diamond_{?}\varphi \coloneqq \diamondcdot\Diamond_{\text{F}}\varphi$.

Given move number x in a chess game α, the operators above may be used to quantify over the games of chess which occupy the same board state as the game α at time x. Letting p_w be the sentence 'The white king is in checkmate', a game of chess may be resigned if the white king will eventually be in checkmate: $\Diamond_{!}p_w$. By contrast, letting p_b be the sentence 'The black king is in checkmate', a game of chess is still worth playing if each player could eventually win which we may express as: $\Diamond_{?}p_b \wedge \Diamond_{?}p_w$.

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[45] The account of the strongest objective modality in **§5.1** derives the **T**, **4**, and **B** axioms in **C1**— **4** from composition, **B** from converse and reversal, and **T** from these together with the consistency of the strongest objective modality— where **K**, **N**, and necessitation hold of every objective modality and so are inherited by $\Box$. The **B** axiom rests on the term-level form of Williamson's principle that every necessity has a reversal: **O-Rev** fixes the converse of each objective modality as its reversal, and **O-Conv** requires that converse to be objective in turn. This helps to justify adopting an **S5** logic for $\Box$ in **TM**.

[46] Since $\langle\tau\rangle_x$ is an equivalence class under the relation $\sigma \sim_x \tau \coloneqq \sigma(x) = \tau(x)$, the monomodal logic of $\boxdot$ is also S5, though $\boxdot$ is strictly weaker than $\Box$ given $\langle\tau\rangle_x \subseteq H_{\mathcal{F}}$, and $\boxdot\varphi \to \triangle\varphi$ is invalid since stability at one time constrains no other. The same logic thus fails to fix an operator's strength or its interaction with tense, which is why $\Box$ is characterized by being strongest among the objective modalities and distinguished from $\boxdot$ by **P1** and **P2**. Moreover, for non-temporal φ, the truth-value of φ at $\langle\mathcal{M}, \tau, x\rangle$ depends only on $\tau(x)$, so $\varphi \to \boxdot\varphi$ is valid, collapsing $\boxdot$ to the trivial modality on this fragment.

Another natural restriction on the set of possible worlds arises from considering only those possible worlds which overlap with a given world up to the present time while possibly diverging in the future, and similarly for the past. More precisely:

Open Futures: $|\tau\rangle_x \coloneqq \{\sigma \in H_{\mathcal{F}} \mid \sigma(y) = \tau(y) \text{ for all } y \leqslant x\}$.

Open Pasts: $\langle\tau|_x \coloneqq \{\sigma \in H_{\mathcal{F}} \mid \sigma(y) = \tau(y) \text{ for all } y \geqslant x\}$.

Whereas $|\tau\rangle_x$ is the set of all possible worlds that occupy the same world state as τ at each time up to and including x while possibly diverging at later times, $\langle\tau|_x$ is the set of all possible worlds that occupy the same world states as τ at x and all later times while possibly diverging at earlier times. Given these definitions, we may introduce operators which quantify over these restricted sets of possible worlds:

($\boxed{\triangleright}$) $\mathcal{M}, \tau, x \vDash \boxed{\triangleright}\varphi$ *iff* $\mathcal{M}, \sigma, x \vDash \varphi$ for all $\sigma \in |\tau\rangle_x$.

($\boxed{\triangleleft}$) $\mathcal{M}, \tau, x \vDash \boxed{\triangleleft}\varphi$ *iff* $\mathcal{M}, \sigma, x \vDash \varphi$ for all $\sigma \in \langle\tau|_x$.

Whereas the *open future* operator $\boxed{\triangleright}$ quantifies over all possible worlds that agree with the world of evaluation up to the time of evaluation, the *open past* operator $\boxed{\triangleleft}$ is also intelligible and has been included for comparison. Although there is much greater occasion to contemplate the range of open futures at a moment while making plans, there is nothing to stop us from also quantifying over the range of open pasts.

Taking possible worlds to be primitive while attempting to provide a semantics for $\boxdot$, $\boxed{\triangleright}$, and $\boxed{\triangleleft}$ requires that frames include primitive accessibility relations $R^x_{\times}$, $R^x_{\triangleright}$, and $R^x_{\triangleleft}$ for each duration $x \in D$. Frame constraints must then be imposed to support the intended behaviors for the corresponding operators. For instance, we might expect $R^x_{\triangleright}$ and $R^x_{\triangleleft}$ to be restrictions of $R^x_{\times}$ with a nonempty intersection, though no such constraints determine a unique extension for each relation. By contrast, the construction of possible worlds makes $\langle\tau\rangle_x$, $|\tau\rangle_x$, and $\langle\tau|_x$ definable, where we may then prove that $|\tau\rangle_x \subseteq \langle\tau\rangle_x$ and $\langle\tau|_x \subseteq \langle\tau\rangle_x$ where $|\tau\rangle_x \cap \langle\tau|_x = \{\tau\}$. Moreover, $|\tau\rangle_y \subseteq |\tau\rangle_x$ and $\langle\tau|_x \subseteq \langle\tau|_y$ for $x \leqslant y$, so that moving forward in time narrows the open futures and widens the open pasts. Whereas the construction of possible worlds from world states, tasks, and durations fixes their relations, taking worlds to be primitive leaves the relations between them to be constrained rather than determined.

Instead of accepting the costs of imposing constraints on a primitive accessibility relation between possible worlds, we may extend the present framework to take the *task relation* to be four-place so that $u \Rightarrow^w_x v$ indicates that it is possible for the world state u to transition to the world state v in duration x relative to the world state w. This extension accommodates laws that are contingent over the task frame or even temporary within a given history. For instance, we may consider a game where the rules may change during the course of the game. Letting $H^w_{\mathcal{F}}$ be the set of possible worlds defined over $\Rightarrow^w$, i.e., the total functions $\tau : D \to W$ where $\tau(x) \Rightarrow^w_{y-x} \tau(y)$ for all $x, y \in D$, we may provide a semantic clause for nomological necessity:

($\boxed{\scriptstyle N}$) $\mathcal{M}, \tau, x \vDash \boxed{\scriptstyle N}\varphi$ *iff* $\mathcal{M}, \sigma, x \vDash \varphi$ for all $\sigma \in H^{\tau(x)}_{\mathcal{F}}$.

Without assuming $\Rightarrow^{w}$ to be invariant with respect to the world states $w \in W$, nothing requires $\boxed{\scriptstyle N}$ to have an S5 logic. By contrast, I will take metaphysical necessity $\Box$ to be the strongest objective modality where an S5 logic is appropriate. Since the present aim is to develop a bimodal logic for tense and metaphysical modality, I will omit further consideration of the restricted modals $\boxdot$, $\boxed{\rhd}$, $\boxed{\lhd}$, and $\boxed{\scriptstyle N}$.

3.2 Bimodal Logic

Recall the propositional language $\mathcal{L} = \langle \mathbb{L}, \bot, \rightarrow, \Box, \lhd, \rhd \rangle$ where $\mathbb{L} := \{p_i \mid i \in \mathbb{N}\}$ is a set of *sentence letters* and the *well-formed sentences* of $\mathcal{L}$ are defined as follows:

$$\varphi,\ \psi \ ::= \ p_i \mid \bot \mid \varphi \rightarrow \psi \mid \Box\varphi \mid \varphi \lhd \psi \mid \varphi \rhd \psi.$$

The *Logic of Tense and Modality* **TM** extends *Classical Propositional Logic* **CPL** to be closed under all instances of the **S5** axiom and rule schemata for the modal $\Box$, the *Burgess–Xu* schemata **BX** for $\lhd$ and $\rhd$, and a single bimodal interaction axiom **MF**, which I will present in turn. The following schemata are familiar from modal logic:

MK $\Box(\varphi \rightarrow \psi) \rightarrow (\Box\varphi \rightarrow \Box\psi)$

MP $\varphi, \varphi \rightarrow \psi \vdash \psi$

MT $\Box\varphi \rightarrow \varphi$

MN *If* $\vdash \varphi$, *then* $\vdash \Box\varphi$

M5 $\Diamond\Box\varphi \rightarrow \Box\varphi$

Letting $\varphi_{\langle \mathrm{S}|\mathrm{U} \rangle}$ be the result of interchanging all occurrences of $\lhd$ and $\rhd$ in φ, the temporal schemata of **BX** are the following, where the past/since direction of each axiom follows from the future/until direction by **TR** below:

TN *If* $\vdash \varphi$, *then* $\vdash \boxed{\scriptstyle F}\varphi$

UI $\varphi \rhd (\varphi \wedge (\varphi \rhd \psi)) \rightarrow \varphi \rhd \psi$

TR *If* $\vdash \varphi$, *then* $\vdash \varphi_{\langle \mathrm{S}|\mathrm{U} \rangle}$

UC $\boxed{\scriptstyle F}(\varphi \rightarrow \psi) \rightarrow ((\chi \rhd \varphi) \rightarrow (\chi \rhd \psi))$

TS $\Diamond_{\mathsf{F}}\top$

UF $(\varphi \rhd \psi) \rightarrow (\varphi \wedge (\varphi \rhd \psi)) \rhd \psi$

TC $\varphi \rightarrow \boxed{\scriptstyle F}\Diamond_{\mathsf{P}}\varphi$

UG $\boxed{\scriptstyle F}(\varphi \rightarrow \chi) \rightarrow ((\varphi \rhd \psi) \rightarrow (\chi \rhd \psi))$

UE $(\varphi \rhd \psi) \rightarrow \Diamond_{\mathsf{F}}\psi$

SU $\theta \wedge (\varphi \rhd \psi) \rightarrow \varphi \rhd (\psi \wedge (\varphi \lhd \theta))$

UT $\Diamond_{\mathsf{F}}\varphi \rightarrow (\top \rhd \varphi)$

NP $\text{Ⓕ}\top \rightarrow \text{Ⓟ}\top$

NA $\text{Ⓕ}\top \rightarrow \boxed{\scriptstyle P}\text{Ⓕ}\top$

NF $\text{Ⓕ}\top \rightarrow \boxed{\scriptstyle F}\text{Ⓕ}\top$

NB $\text{Ⓕ}\top \rightarrow \Box\text{Ⓕ}\top$

TL $(\Diamond_{\mathsf{F}}\varphi \wedge \Diamond_{\mathsf{F}}\psi) \rightarrow [\Diamond_{\mathsf{F}}(\Diamond_{\mathsf{F}}\varphi \wedge \psi) \vee \Diamond_{\mathsf{F}}(\varphi \wedge \psi) \vee \Diamond_{\mathsf{F}}(\varphi \wedge \Diamond_{\mathsf{F}}\psi)]$

CN $[(\varphi \rhd \psi) \wedge (\chi \rhd \theta)] \rightarrow [(\varphi \wedge \chi) \rhd (\psi \wedge \theta) \vee (\varphi \wedge \chi) \rhd (\psi \wedge \chi) \vee (\varphi \wedge \chi) \rhd (\varphi \wedge \theta)]$

Whereas **TR** makes the logic symmetric with respect to the past and future at each time, **TS** asserts that every time has a strictly later time, guaranteeing seriality where the past dual follows by **TR**. Whereas **TC** asserts that the present is past to all future times, **TL** asserts that any two future times are linearly ordered. Given **TR**, we may also conclude that the present is future to all past times and any two past times are linearly ordered from which it follows that time is linear. The schemata **UE** – **SU** govern $\rhd$ directly, relating it to the defined operators $\boxed{\scriptstyle\text{F}}$ and $\Diamond\kern-0.55em{\scriptstyle\text{F}}$, where **CN** requires any two $\rhd$-claims to be jointly witnessed in one of three ways. Finally, the uniformity axioms require any time with an immediate successor to have an immediate predecessor by **NP**, where **NF**, **NA**, and **NB** require every future time, past time, and possible world to be alike in this respect. Given that $\textcircled{\scriptsize F}\top$ is equivalent to $\bot$ at any time with no immediate successor, these axioms hold vacuously unless time is discrete. The soundness and completeness of **TM** given in **§5.4** and **C6** of the *Appendix* are verified in the Lean 4 repository for this paper.[47]

Whereas the axioms and rules discussed so far include either modal or temporal operators, **TM** includes a single bimodal interaction axiom asserting that what is necessary is necessarily always going to be the case:

MF $\Box\varphi \to \Box\boxed{\scriptstyle\text{F}}\varphi$.

Since the task relation is indexed by durations, the *Appendix* shows in **L17** that $H_{\mathcal{F}}$ is closed under translation. Given that the sentence letters are interpreted by sets of world states rather than world-time pairs, **L18** shows that translation preserves truth, so that whatever is true in τ at x is true at y in the translate of τ by $y - x$. Thus **T17** proves that if φ is true in every possible world at x, then φ is true in every possible world at each later time y, since each possible world is the translate by $y - x$ of a possible world in which φ is true at x. Whereas *Abundance* had to be imposed on the models of $\mathcal{L}^{\text{K}}$ in **§2.3**, the translates of a possible world are possible worlds by construction, and so **MF** answers to no constraint on task frames beyond indexing the task relation by durations rather than fixed times.[48]

Given **MF**, we may derive a number of perpetuity principles. Since $\Box\boxed{\scriptstyle\text{F}}\varphi \to \boxed{\scriptstyle\text{F}}\varphi$ is an instance of **MT**, $\Box\varphi \to \boxed{\scriptstyle\text{F}}\varphi$ follows from **MF** where $\Box\varphi \to \boxed{\scriptstyle\text{P}}\varphi$ follows by **TR**. Together with **MT**, these results entail $\Box\varphi \to (\boxed{\scriptstyle\text{P}}\varphi \wedge \varphi \wedge \boxed{\scriptstyle\text{F}}\varphi)$ which is equivalent to:

P1 $\Box\varphi \to \triangle\varphi$. **P2** $\triangledown\varphi \to \Diamond\varphi$.

This derivation shows that the perpetuity principles follow from **MF** and **MT** by classical reasoning. In addition to being stated in primitive terms, **MF** and **MT** are easy to justify. We may also derive the following principle:

TF $\Box\varphi \to \boxed{\scriptstyle\text{F}}\Box\varphi$.

Whereas $\Box\varphi \to \Box\Box\varphi$ follows by standard modal reasoning, $\Box\Box\varphi \to \Box\boxed{\scriptstyle\text{F}}\Box\varphi$ and $\Box\boxed{\scriptstyle\text{F}}\Box\varphi \to \boxed{\scriptstyle\text{F}}\Box\varphi$ are instances of **MF** and **MT**. Composing yields $\Box\varphi \to \boxed{\scriptstyle\text{F}}\Box\varphi$ where $\Box\varphi \to \boxed{\scriptstyle\text{P}}\Box\varphi$ follows by **TR**. We may then derive $\Box\varphi \to (\Box\boxed{\scriptstyle\text{P}}\varphi \wedge \Box\varphi \wedge \Box\boxed{\scriptstyle\text{F}}\varphi)$ where $\Box\varphi \to \Box(\boxed{\scriptstyle\text{P}}\varphi \wedge \varphi \wedge \boxed{\scriptstyle\text{F}}\varphi)$ follows by modal reasoning. Given the definitions, we have:

[47] See the repository for the Lean 4 formalization.

[48] Were the task relation indexed by times rather than durations, so that the translate of a possible world need not be a possible world, **MF** would instead require each sentence $\Box\varphi$ to be true at all times or at none, since given the **S5** schemata its instances for $\Box\varphi$ and $\neg\Box\varphi$ make the sets of times at which each is true upward closed, and a total order cannot be partitioned into two nonempty upward closed sets.

P3 $\Box\varphi \to \Box\triangle\varphi$. **P4** $\Diamond\triangledown\varphi \to \Diamond\varphi$.

Since the distribution principle **TK** and the transitivity principle **T4** are theorems of **BX** by standard modal reasoning, they may be used freely in what follows:

TK $\boxed{\mathrm{F}}(\varphi \to \psi) \to (\boxed{\mathrm{F}}\varphi \to \boxed{\mathrm{F}}\psi)$. **T4** $\boxed{\mathrm{F}}\varphi \to \boxed{\mathrm{F}}\boxed{\mathrm{F}}\varphi$.

Since $\Box\Diamond\varphi \to \Diamond\varphi$ is an instance of **MT**, it follows from **TK** that $\boxed{\mathrm{F}}\Box\Diamond\varphi \to \boxed{\mathrm{F}}\Diamond\varphi$. We may then derive $\Diamond\varphi \to \Box\Diamond\varphi$ from **M5**, where $\Box\Diamond\varphi \to \boxed{\mathrm{F}}\Box\Diamond\varphi$ is an instance of **TF**. Thus $\Diamond\varphi \to \boxed{\mathrm{F}}\Diamond\varphi$ where $\Diamond\varphi \to \boxed{\mathrm{P}}\Diamond\varphi$ follows by **TR**, and so $\Diamond\varphi \to (\boxed{\mathrm{P}}\Diamond\varphi \wedge \Diamond\varphi \wedge \boxed{\mathrm{F}}\Diamond\varphi)$ which is equivalent to $\Diamond\varphi \to \triangle\Diamond\varphi$. Given **P4**, we may derive the following:

P5 $\Diamond\triangledown\varphi \to \triangle\Diamond\varphi$. **P6** $\triangledown\Box\varphi \to \Box\triangle\varphi$.

The perpetuity principles **P1** – **P6** begin to characterize the interactions between tense and modality in **TM**, providing abductive support for the logic. The remainder of this section strengthens **TM** by presenting three complete extensions alongside their corresponding frame constraints, before considering an extension to the language.

3.3 Extensions

Since the logic for metaphysical modality already describes the strongest objective modality, it remains to strengthen the tense logic by constraining the temporal order $\mathcal{D}$. In addition to taking $\mathcal{D}$ to be a nontrivial totally ordered abelian group, consider:

Discrete: For any time $x \in D$, if there is a later time $y > x$, then there is a least later time $y' > x$ where $z \geqslant y'$ for all $z > x$.[49]

Dense: For any times $x, y \in D$ where $x < y$, there is a time $z \in D$ where $x < z < y$.

Complete: Every nonempty set $X \subseteq D$ bounded above has a least upper bound.

Each frame constraint places a restriction on $\mathcal{D} = \langle D, +, 0, \leqslant\rangle$, and corresponds to a schema of $\mathcal{L}$ which is valid over every task frame with a given temporal order just in case that order satisfies the constraint, as shown in **T10**, **T11**, and **T12**, respectively:

DF $(\boxed{\mathrm{P}}\varphi \wedge \varphi \wedge \Diamond_{\mathrm{F}}\top) \to \Diamond_{\mathrm{F}}\boxed{\mathrm{P}}\varphi$.

DN $\boxed{\mathrm{F}}\boxed{\mathrm{F}}\varphi \to \boxed{\mathrm{F}}\varphi$.

CO $\triangle(\boxed{\mathrm{P}}\varphi \to \Diamond_{\mathrm{F}}\boxed{\mathrm{P}}\varphi) \to (\boxed{\mathrm{P}}\varphi \to \boxed{\mathrm{F}}\varphi)$.

Along a possible world on which no truth value varies, $\boxed{\mathrm{P}}\varphi$ and $\boxed{\mathrm{F}}\varphi$ are equivalent to φ, and so each correspondent above is valid over any task frame whose possible worlds are all constant, whatever its temporal order may be. As a result, the correspondents above fail to define the corresponding class of task frames.

[49] It follows that if there is an earlier time $y < x$, then there is a greatest earlier time. Since $\mathcal{D}$ is an ordered group, the map $x \mapsto -x$ reverses the order. Given $y < x$, we have $-x < -y$. The Discrete condition provides an earliest time $z > -x$, and so $-z < x$ is the latest time before x.

Consider first the discrete orders. Although **DF** provides an immediate successor at every time, it is valid over every discrete order, including those in which a later time cannot be reached from an earlier one in finitely many steps, and so it does not distinguish $\mathbb{Z}$-time, over which completeness is sought, from the discrete orders at large. The *discrete extension* $\mathbf{TM}_{\mathrm{Z}}$ instead adds two principles to **TM**:

UZ $\quad \diamondplus\varphi \to (\neg\varphi \rhd \varphi)$.

Z1 $\quad \boxed{\scriptstyle F}(\boxed{\scriptstyle F}\varphi \to \varphi) \to (\diamondplus\boxed{\scriptstyle F}\varphi \to \boxed{\scriptstyle F}\varphi)$.

Since **UZ** requires every future φ-time to be preceded by a *nearest* one, its instance for $\top$ delivers the immediate successor that **DF** demands. The backward induction principle **Z1** requires every later time to be reached in finitely many steps, narrowing the discrete orders to $\mathbb{Z}$-time. Whereas **UZ** and **Z1** make **DF** derivable, the *dense extension* $\mathbf{TM}_{\mathrm{D}}$ retains **DN** while adding the following principle:

NN $\quad \neg\textcircled{\scriptsize F}\top$.

In addition to imposing **DN** which requires some time to intervene between the present and any later time, **NN** denies that there is an immediate successor.[50] In contrast, **CO** concerns endpoints, and is derivable in the *dense and complete extension* $\mathbf{TM}_{\mathrm{R}}$, which adds two principles from Reynolds' axiomatization [38, §2] to $\mathbf{TM}_{\mathrm{D}}$. Letting $K^{+}\varphi \coloneqq \neg(\neg\varphi \rhd \top)$ and $K^{-}\varphi \coloneqq \neg(\neg\varphi \lhd \top)$ assert that φ recurs arbitrarily soon in the future and recurred arbitrarily recently in the past, consider the principles:

PU $\quad [(\varphi \rhd \top) \wedge \diamondplus\neg\varphi] \to [\varphi \rhd (\neg\varphi \vee K^{+}\neg\varphi)]$.

SEP $\quad K^{+}\varphi \wedge \neg K^{+}[\varphi \wedge (\neg\varphi \rhd \varphi)] \to K^{+}(K^{+}\varphi \wedge K^{-}\varphi)$.

Whereas **CO** requires a region defined by $\boxed{\scriptstyle P}$ to have an endpoint, **PU** requires the same of a region defined by $\rhd$, and **SEP** is a separation principle which owes its validity over $\mathbb{R}$-time to the separability of $\mathbb{R}$.[51] Since **PU** alone suffices to derive **CO** in **TM**, the latter is omitted from the axioms of **D38**, where only the future direction of **PU** is stated since its past dual follows by **TR**. Since $\diamondplus\top$ is **TS** and $\neg\top \rhd \top$ is $\textcircled{\scriptsize F}\top$, the instance of **UZ** for $\top$ contradicts **NN**, and so no consistent extension of **TM** includes both, just as no temporal order is both discrete and dense. Since every dense and complete order is dense, $\mathbf{TM}_{\mathrm{R}}$ includes $\mathbf{TM}_{\mathrm{D}}$.

As shown in **C6**, $\mathbf{TM}_{\mathrm{D}}$ is strongly complete over the dense task frames, and $\mathbf{TM}_{\mathrm{Z}}$ and $\mathbf{TM}_{\mathrm{R}}$ are weakly complete over $\mathbb{Z}$-time and $\mathbb{R}$-time, respectively.[52] Whereas **DF** is valid over every discrete temporal order, **UZ** and **Z1** are valid only over the

[50] In the static task frame over $\mathbb{Z}$ described in the footnote to **T10**, every possible world is constant, so $\boxed{\scriptstyle F}\varphi$ is equivalent to φ and every instance of **DN** is valid, yet $\textcircled{\scriptsize F}\top$ is true at every time since $\mathbb{Z}$ is discrete. Since **TM** is sound over every task frame, its extension by **DN** alone does not prove **NN**.

[51] **PU** is Reynolds' [38, §2] Prior-U. **SEP** is due to Gabbay and Hodkinson [39], given here in the form given in Gabbay, Hodkinson, and Reynolds [40, §6.9]. Its validity over $\mathbb{R}$-time is proved in Reynolds [38, §7].

[52] A temporal order is *Archimedean* if for any $x, y \in D$ where $x > 0$, there is a positive integer n where $nx > y$. A discrete temporal order is Archimedean just in case every duration is bounded by some multiple $n \cdot 1$ of its least positive duration 1. Every nontrivial Complete totally ordered group is Archimedean, and by Hölder's theorem [41] every Archimedean totally ordered group embeds in $\mathbb{R}$. Since a nontrivial subgroup of $\mathbb{R}$ is cyclic if discrete and all of $\mathbb{R}$ if dense and complete, the discrete Archimedean orders are exactly $\mathbb{Z}$-time, the dense and complete orders exactly $\mathbb{R}$-time, and the complete orders $\mathbb{Z}$ or $\mathbb{R}$ up to isomorphism.

Archimedean discrete orders (**P3**), and these are exactly $\mathbb{Z}$-time. It follows that $\mathbf{TM}_{\mathbb{Z}}$ is the logic of $\mathbb{Z}$-time rather than of the discrete task frames. Since there is little sense in arguing for one logic over another independent of an application, the choice between the extensions of **TM** turns on the dynamical systems under study.

For any task frame $\mathcal{F}$, the semantic clauses for the metaphysical modals quantify over all possible worlds in $H_{\mathcal{F}}$ which are defined over the temporal order $\mathcal{D}$ included in $\mathcal{F}$. As a result, the structure of $\mathcal{D}$— discrete or dense, and either Dedekind complete or not— holds of metaphysical necessity for that system. For instance, if $\mathcal{D}$ is dense, then **DN** and its necessitation are valid over this task frame. That density is necessary if true is not a special feature of the task semantics but an instance of the fact that frame validity is closed under necessitation: whatever holds at every point of every model on a task frame is necessary there, since $\Box$ quantifies only over that task frame's possible worlds. One might object that the present account overreaches, claiming instead that whether time is discrete or dense is a contingent matter.[53] But truth is only evaluated relative to a model which is defined over a frame. There is no modality quantifying over possible worlds drawn from distinct task frames.[54] Disputes about the structure of time concern which logic and corresponding class of task frames is appropriate for the dynamical systems under study, not what is metaphysically contingent *simpliciter*. Kripke semantics supplies the precedent: over the class of symmetric frames, the **B** axiom and its necessitation are both valid, and so symmetry is necessary if true. Structural disputes about metaphysical modality already take this form: whether an **S4** or **S5** logic is correct for $\Box$ is debated as a question of which logic and class of frames are correct, not the claim that symmetry is itself metaphysically contingent.[55] The correspondence-theoretic methodology assumed in **§2.3** takes a frame constraint paired with a proved correspondence theorem to be the mark of successful semantics, not a liability calling for repair. It is an advantage of the present framework that such disputes can be articulated in $\mathcal{L}$ without further expressive resources.[56]

Besides extending **TM**, one may also extend $\mathcal{L}$ itself. Although $\lhd$ and $\rhd$ afford considerable expressive power, **TM** and its extensions lack the means by which to cross reference either times or worlds. To lend greater discriminating power to the language, we may include a vector $\vec{v} = \langle v_1, v_2, \ldots\rangle$ of stored times and vector $\vec{\mu} = \langle \mu_1, \mu_2, \ldots\rangle$ of

[53] Dorr and Goodman [30, p. 656] express sympathy for an account of metaphysical modality that is able to assert the contingency of the structure of time. To give voice to such contingency in this framework, one might admit *irregular worlds*— functions $\tau : X \to W$ where $X \subsetneq D$ is a *coset domain*, a translate $G + c$ of a nontrivial subgroup $G \leqslant \mathcal{D}$, and $\tau(x) \Rightarrow_{y-x} \tau(y)$ for all $x, y \in X$— and defining consequence over the irregular and possible worlds alike. Since a family of translates is closed under ambient translation, **MF** and the perpetuity principles are preserved. The price is exact: every nontrivial ordered abelian group contains a discrete cyclic subgroup, so **DN** is then valid over no task frame whatever, and **DF** fails over discrete orders possessing a subgroup that is itself dense, such as $\mathbb{Q} \times_{\text{lex}} \mathbb{Z}$, the pairs $\langle q, n\rangle$ ordered by the first coordinate and by the second only when the first agrees, so the correspondence results of **T10**, **T11**, and **T12** collapse together. These considerations recommend possible over irregular worlds.

[54] A reader who takes metaphysical modality to concern reality taken as a whole may recover absoluteness within the present framework by taking the system under study to be the universal one— the *cosmos* in its full modal extent— where the intended task frame includes all world states and durations whatsoever.

[55] The *Appendix* **5.1** takes the objective modalities to be closed under converses to derive **C1**, providing grounds for thinking that it is an analytic truth that metaphysical modality has an **S5** logic.

[56] Over the unrestricted class of task frames, **DF**, **DN**, and **CO** are not valid, nor are their negations: it is not *analytic*— encoded as *truth in all models*— that time is discrete, dense, or complete, nor that it fails to be. By contrast, **TL** and its necessitation are valid over the class of dense task frames, and so it is analytic that time is linear in the present framework in much the same way that identity is reflexive.

stored worlds in the point of evaluation to provide the semantic clauses for the *store operators* $\uparrow^i_{\mathrm{T}}$ and $\uparrow^i_{\mathrm{M}}$ and *recall operator* $\downarrow^i_{\mathrm{T}}$ and $\downarrow^i_{\mathrm{M}}$ for each $i \in \mathbb{N}$:

($\uparrow_{\mathrm{T}}$) $\mathcal{M}, \tau, x, \vec{v}, \vec{\mu} \vDash \uparrow^i_{\mathrm{T}}\varphi$ *iff* $\mathcal{M}, \tau, x, \vec{v}_{[x/v_i]}, \vec{\mu} \vDash \varphi$.

($\downarrow_{\mathrm{T}}$) $\mathcal{M}, \tau, x, \vec{v}, \vec{\mu} \vDash \downarrow^i_{\mathrm{T}}\varphi$ *iff* $\mathcal{M}, \tau, v_i, \vec{v}, \vec{\mu} \vDash \varphi$.

($\uparrow_{\mathrm{M}}$) $\mathcal{M}, \tau, x, \vec{v}, \vec{\mu} \vDash \uparrow^i_{\mathrm{M}}\varphi$ *iff* $\mathcal{M}, \tau, x, \vec{v}, \vec{\mu}_{[\tau/\mu_i]} \vDash \varphi$.

($\downarrow_{\mathrm{M}}$) $\mathcal{M}, \tau, x, \vec{v}, \vec{\mu} \vDash \downarrow^i_{\mathrm{M}}\varphi$ *iff* $\mathcal{M}, \mu_i, x, \vec{v}, \vec{\mu} \vDash \varphi$.

Whereas $\uparrow^i_{\mathrm{T}}\varphi$ replaces the i^{th} value of $\vec{v}$ with the evaluation time and $\uparrow^i_{\mathrm{M}}\varphi$ replaces the i^{th} value of $\vec{\mu}$ with the evaluation world, $\downarrow^i_{\mathrm{T}}\varphi$ shifts the evaluation time to the i^{th} value stored in $\vec{v}$ and $\downarrow^i_{\mathrm{M}}\varphi$ shifts the evaluation world to the i^{th} value stored in $\vec{\mu}$. These operators generalize the *now* and *then* operators that Vlach [42] introduces to resolve tense anaphora, replacing Vlach's single pair of operators with an indexed family $\uparrow^i_{\mathrm{T}}$ and $\downarrow^i_{\mathrm{T}}$ as well as providing their world-theoretic counterparts $\uparrow^i_{\mathrm{M}}$ and $\downarrow^i_{\mathrm{M}}$ to store and recall arbitrarily many times and worlds. Despite adding $\vec{v}$ and $\vec{\mu}$ as parameters to the point of evaluation, the semantics for $\mathcal{L}$ may otherwise be maintained. Although extending **TM** to provide a logic for the resulting language is outside the scope of the present paper, the following section will employ the semantics above to address the paradox of the open future and to define the Deterministic task frames using the time store and recall operators which cannot be done by extending $\mathcal{L}$ to include $\boxdot$.

4 Tense and Modality

The construction of possible worlds assumes that time is a total order. For any world $\tau \in H_{\mathcal{F}}$ and time $x \in D$ in a model $\mathcal{M}$ of $\mathcal{L}$, there is a determinate past and future relative to x where every well-formed sentence of $\mathcal{L}$ has a unique truth-value in τ at x. However natural it may be to take the past to be determined up to any given moment, it is much more contentious to suppose that each time determines a unique future. Arguments as old as Aristotle have sought to lend credence to the idea that the future differs from the past on account of remaining open to determination by admitting a range of incompatible future alternatives, none of which is singled out as the actual future. Thomason [26] brings this point out as follows:

> [T]he basic issue here seems to be whether or not one is prepared to accept as meaningful the assertion that there is always, whether we know it or not, a single possible future which, from the perspective of a given time will be its actual future. (p. 270)

In considering future contingents such as 'There will be a sea battle tomorrow', many have sought to follow Aristotle in denying that such claims have a truth-value until the future comes about. Although there may be no fact of the matter whether there will be a sea battle tomorrow when it is considered today, we may nevertheless expect it to be settled tomorrow whether there ends up being a sea battle or not.

The construction of possible worlds takes each time in each possible world to have a complete past and future that is determined in every respect. However, evaluating a future contingent claim at a possible world and time does not carry any commitment that the world of evaluation is the actual world whose future is guaranteed to take place. By considering all worlds that intersect the world of evaluation at the time of evaluation, one may leave it open which of those worlds will play out. Just as one may consider the various chess games that may transpire from a given board state, it is also natural to consider the various routes between two points on a map, the different states a computer may transition through, or the different sequences of actions by one or more agents, where each of these may be represented by the possible worlds in one system or another. Despite making evaluations according to one possible world or another when considering the evolution of a system, doing so does not require any of the possible worlds to be the *actual* world that will take place. Even if none of the possible worlds are foretold to be actual, we can say in full confidence what each possible world specifies. The following subsection will defend a deflationary approach of this kind by considering the shortcomings that face the alternatives.

4.1 Open Future

Dynamical systems of all kinds admit an open future. Chess continues to serve as a paradigm case. If each board state determined all future board states, there would only be one game of chess to rehearse, leaving little semblance of a game worth playing. Just as the moves in chess narrow the range of open futures by determining what future board state becomes actual, human actions within non-deterministic systems carry consequences. When I choose soup over salad, I determine the next course of my meal, ruling out the futures in which salad precedes the main course while leaving open futures in which I have dessert as well as those in which I do not.

By formally describing both deterministic and non-deterministic systems, we may ask as a strictly empirical matter whether any given system is deterministic or not. Whereas dynamical systems theory provides resources for modeling the evolution of a given system, **TM** provides logical resources for reasoning about those systems in the bimodal object language $\mathcal{L}$. In particular, consider the following axiom schema:

Determined: $\varphi \rightarrow \boxdot\varphi$.

In any deterministic system, all instances of the schema above are true. For instance, if determinism is true of our world and I am going to visit Ladakh, then I will eventually visit Ladakh, or in symbols: $\Diamond_{F} p_l \rightarrow \boxed{!}p_l$. Consider the corresponding frame constraint:

Deterministic: For $w, u, v \in W$ and $x \in D$, if $w \Rightarrow_x u$ and $w \Rightarrow_x v$, then $u = v$.

Whereas the frame constraints given in **§3.3** impose a restriction on $\mathcal{D}$, the constraint above articulates what it is for the task relation $\Rightarrow$ to be deterministic. Over any task frame satisfying Deterministic, exactly one possible world passes through any given world state at any given time, and so $\langle\tau\rangle_x = \{\tau\}$ for every $\tau \in H_{\mathcal{F}}$ and $x \in D$ by **L19**. Since $\boxdot\varphi$ requires φ to hold in every world through the current world state at the current time and there is only the current world to consider, *Determined* is valid as **T13** shows. Because $\boxdot\varphi \rightarrow \varphi$ is valid over every task frame, *Determined* holds over a task frame just in case the stability operator is idle there. Nevertheless, the idleness of the stability operator does not suffice for determinism. The drift frame $\mathcal{F}^{\circ}$

of **T14** is not Deterministic and yet validates every instance of *Determined*, and so *Determined* does not define the Deterministic task frames.[57]

The present approach distinguishes between deterministic and non-deterministic systems independent of temporal structure. Thus **TL** may be retained, maintaining compatibility with the view that it is analytic that time is linear. Within each possible world $\tau \in H_{\mathcal{F}}$, time is a total order where every sentence receives a determinate truth-value at each time in τ. The openness of the future consists not in the existence of incomparable future times but in the absence of any possible world which is designated as the *actual* world: different possible worlds may pass through the same world state, where none is singled out as the one that has and will continue to obtain. This stands in contrast to the proposal to accommodate an open future by dropping **TL** and replacing the linear temporal order with a partial order. The remainder of this section considers the difficulties that branching-time faces and explains how the task semantics achieves a superior result while retaining a linear theory of time.

Instead of requiring the times in each task frame to be totally ordered as in **§3**, there is the tradition following Prior [22] and Kripke [21] that weakens the tense logic by dropping **TL** and taking the times to form a connected left-linear partial order $\langle T, < \rangle$ so that each time $x \in T$ may admit incomparable futures.[58] Despite admitting branching times, one might nevertheless attempt to preserve the definition from **§3**. Since branching times cannot retain the group structure of $\mathcal{D}$, durations must be supplied separately by labeling each pair of comparable times $x < y$ with a duration $d(x, y) \geqslant 0$ that is additive along chains. A partial history is then any $\tau : X \to W$ with $X \subseteq T$ where $\tau(x) \Rightarrow_{d(x,y)} \tau(y)$ for all comparable $x < y$ in X, requiring X to be convex for convex histories and total for possible worlds. However, this proposal faces the same difficulties that arise for the Peircean semantics considered in **§2.1**. For instance, consider a chess game τ at a time x in which there is a future time $y > x$ where the black king is in checkmate and an incomparable future time $z > x$ where the white king is in checkmate. By the semantics for the future operator $\Diamond_{\mathrm{F}}$, 'White is going to win' and 'Black is going to win' are both true, though this would seem impossible. It is much more natural to assert 'White *could eventually* win' and 'Black *could eventually* win' to indicate that there is some future in which White wins and some future in which Black wins, where these are the sorts of chess games that are still worth playing. Such a theorist might take $\Diamond_{\mathrm{F}}$ to read 'could eventually' which requires φ at some later time in some branch, and take $\boxed{\mathrm{F}}$ to read 'will always' which requires φ at all later times across all branches. Although this reading is more fitting here, a branching-time theory of this kind inherits the expressive limitations of the Peircean semantics. For instance, there is no way to express the density of time, insofar as density $\boxed{\mathrm{F}}\boxed{\mathrm{F}}\varphi \to \boxed{\mathrm{F}}\varphi$ concerns the structure of any one branch rather than all future times distributed over all branches. Following the Ockhamist by introducing a further parameter at which to evaluate sentences that specifies the particular branch

[57] No strengthening of *Determined* in $\mathcal{L}$ extended with $\boxdot$ fares better, since **C4** shows that $\mathcal{F}^{\circ}$ validates exactly the sentences of that language validated by the Deterministic translation frame $\mathcal{F}^{1}$. Even so, extending **TM** by *Determined* together with axioms governing $\boxdot$ yields a system sound over the task frames validating *Determined* and complete over the Deterministic task frames, and so the two classes share a logic in that language, as established in the Lean 4 repository for this paper.

[58] One must also give up **TR** to validate $\Diamond_{\mathrm{F}}\top \to (\boxed{\mathrm{P}}\boxed{\mathrm{F}}\varphi \to \triangle\varphi)$ which requires the past to be unique and then add the result of interchanging $\boxed{\mathrm{P}}$ and $\boxed{\mathrm{F}}$ to recover what is otherwise derivable from **TR**.

within a possible world returns the situation that we were in before where sentences are evaluated at parameters that suffice to determine a fixed past and future.

Consider a game of chess in which both Black and White have a chance at winning after 14 moves have been played. Given the state of the board, there simply is no actual game with a determinate future that proceeds from the present board state. Rather, we may consider various chess games that proceed from the present board state after move 14. By taking turns choosing which moves to make, the players compete in their abilities to determine which game of chess becomes actual. Instead of building indeterminacy into the temporal structure itself by taking time to be only partially ordered, the task semantics for $\mathcal{L}$ locates indeterminacy in the task relation between world states over a duration so that both $w \Rightarrow_x u$ and $w \Rightarrow_x v$ for world states $u \neq v$ and duration $x > 0$. Letting time be a total order, each possible world $\tau \in H_{\mathcal{F}}$ represents one possible path through the space of world states, constrained by the transitions which the task relation permits. Different possible worlds intersecting the same world state represent genuine alternatives about how the system could evolve without requiring multiple incomparable future times. So long as none of these possible worlds are assumed to be actual, no harm comes in allowing each possible world to specify a determinate past and future from any given world state at a time.

Something similar may be said for Aristotle's sea battle. Since some of the possible worlds intersecting the present world state include a sea battle on the following day and other intersecting worlds do not, it is contingent whether there is going to be a sea battle. In symbols, one might assert $\Diamond\!\!\!\!\;{\scriptstyle ?}\,\neg p_b \wedge \Diamond\!\!\!\!\;{\scriptstyle ?}\, p_b$ where p_b expresses that a sea battle is taking place and $\Diamond\!\!\!\!\;{\scriptstyle ?}\,\varphi \coloneqq \Diamond\!\!\!\!\;\cdot\;\Diamond\!\!\!\!\;{\scriptscriptstyle F}\,\varphi$ as before, though this is too easy to satisfy.[59] Given a possible world in a deterministic task frame where the ships first clash at 2pm tomorrow, both $\Diamond\!\!\!\!\;{\scriptstyle ?}\,\neg p_b$ and $\Diamond\!\!\!\!\;{\scriptstyle ?}\, p_b$ could be true as witnessed by the time 11am while the ships were still yet to meet and 2:30pm after the battle started. Consider instead:

$$\uparrow^1_{\mathrm{T}} \Diamond\!\!\!\!\;{\scriptscriptstyle F} \uparrow^2_{\mathrm{T}} \downarrow^1_{\mathrm{T}} (\Diamond\!\!\!\!\;\cdot\, \downarrow^2_{\mathrm{T}} \neg p_b \wedge \Diamond\!\!\!\!\;\cdot\, \downarrow^2_{\mathrm{T}} p_b) \tag{Sea}$$

The claim above asserts that there is a future time where p_b is contingent on account of being true in one possible future and false in another. More generally, we may consider the following schema which is invalid given the truth of the claim above:

$$\uparrow^1_{\mathrm{T}} \boxed{\scriptstyle F} \uparrow^2_{\mathrm{T}} \downarrow^1_{\mathrm{T}} (\boxdot \downarrow^2_{\mathrm{T}} \neg\varphi \vee \boxdot \downarrow^2_{\mathrm{T}} \varphi) \tag{Det}$$

For a given future time, the possible worlds intersecting the present need not agree on φ at that time. In particular, during a day of strategic uncertainty, it may be indeterminate whether there will be a sea battle tomorrow on account of possible worlds that intersect with the present in which there is a sea battle twenty-four hours from now as well as intersecting possible worlds in which there is not.[60]

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[59] Given that $\Diamond\!\!\!\!\;{\scriptstyle ?}\,\neg p_b \wedge \Diamond\!\!\!\!\;{\scriptstyle ?}\, p_b$ is satisfiable, $\boxed{!}\varphi \vee \boxed{!}\neg\varphi$ is invalid despite retaining **TL** and **TR**. However, invalidity is had too easily, since we need only consider a single world in which there are future times where φ and other future times where $\neg\varphi$. Something similar may be said for the satisfiability of $\Diamond\!\!\!\!\;{\scriptstyle ?}\,\neg p_b \wedge \Diamond\!\!\!\!\;{\scriptstyle ?}\, p_b$.

[60] Cariani and Santorio [43] treat 'will' as a non-quantificational modal governed by a selection function rather than a universal quantifier over worlds. To retain bivalence and reject MacFarlane's [44] relativized truth, the indeterminacy of future contingents is located in facts about a plurality of live alternatives rather than in truth-value gaps. The present proposal is more deflationary still, since no possible world with a complete past and future is designated actual— even indeterminately— while still validating classical logic.

Despite entailing indeterminacy, Sea does not provide a natural regimentation of the claim that there will be a sea battle tomorrow by locating the contingency in question at a single future time. The claim does not concern any particular moment— twenty-four hours from now, or otherwise— but rather the temporal interval spanning tomorrow's daylight. The operators $\lhd$ and $\rhd$ of $\mathcal{L}$ permit a better regimentation than $\boxed{\scriptstyle P}$ and $\boxed{\scriptstyle F}$ can express, given the following definitions:

After: $\mathsf{A}(\varphi, \psi) \coloneqq \neg\varphi \rhd (\varphi \wedge \psi)$.

Release: $\mathsf{R}(\varphi, \psi) \coloneqq \neg(\neg\varphi \rhd \neg\psi)$.

Before: $\mathsf{B}(\varphi, \psi) \coloneqq \mathsf{R}(\psi, \neg\varphi) \wedge \Diamond\varphi$.

Given the clause for $\rhd$, $\mathsf{A}(\varphi, \psi)$ is true in τ at x just in case φ and ψ are both true at some time $z > x$ where φ fails at every time y for which $x < y < z$, and so reads 'when φ next obtains, ψ'. Similarly, $\varphi\mathsf{R}\psi$ is true in τ at x just in case, for every time $z > x$, either ψ is true at z or φ is true at some time y for which $x < y < z$, and so reads 'ψ holds until released by φ'. Finally, $\mathsf{B}(\varphi, \psi)$ is true in τ at x just in case φ is true at some time $z > x$ and, whenever φ is true at a time $z > x$, ψ is true at some time y for which $x < y < z$, and so reads 'ψ occurs before φ next does'.[61]

Both A and B are future-directed and target the next occurrence of φ. Letting p_r and p_s express that the sun rises and that the sun sets respectively, we may regiment the claim that a sea battle takes place tomorrow as $\mathsf{A}(p_r, \mathsf{B}(p_s, p_b))$, asserting that a sea battle occurs after the next sunrise but before the next sunset. The sea battle's contingency may then be expressed as follows:

$$\Diamond\mathsf{A}(p_r, \mathsf{B}(p_s, p_b)) \wedge \Diamond\mathsf{A}(p_r, \neg\mathsf{B}(p_s, p_b)) \tag{Sea$'$}$$

Given the operators defined above, Sea$'$ locates the contingency of the sea battle in an event-bounded interval, requiring intersecting worlds to disagree about whether a battle falls between tomorrow's sunrise and sunset.[62] Neither ranging over all future times as the tense operators do nor fixing a single stored time as Sea does, this guarded quantification exhibits the expressive power that $\lhd$ and $\rhd$ lend to $\mathcal{L}$.

As **T15** shows, Det is valid over any deterministic task frame.[63] By contrast, it is easy to conceive of non-deterministic systems in which Det is false where Sea requires this to be the case. Given a possible world $\tau \in H_{\mathcal{F}}$, time $x \in D$, and stored times $\vec{v}$

[61] The operator A is the strong form of Kröger's [45] *atnext* operator: $\mathsf{A}(\varphi, \psi)$ requires ψ to hold *at the next* time φ obtains. Likewise R is the *release* operator of LTL [36], the dual of $\rhd$, and $\mathsf{B}(\varphi, \psi)$ is its strong form, where the conjunct $\Diamond\varphi$ discharges the 'or else forever' disjunct in $\psi\mathsf{R}\neg\varphi$, and where the *precedes* operator of Manna and Pnueli [46] shares the same negated-until shape. Since $\textcircled{\scriptsize F}\psi$ is equivalent to $\mathsf{A}(\top, \psi)$, it follows that A generalizes $\textcircled{\scriptsize F}$. The past-directed readings of 'after' and 'before' are definable by interchanging $\rhd$ and $\lhd$, making $\textcircled{\scriptsize P}$ definable by similar means, so that $\textcircled{\scriptsize P}\varphi$ is equivalent to the past-directed version $\mathsf{A}^{-}(\top, \varphi)$.

[62] The claim Sea$'$ presupposes $\boxdot\mathsf{A}(p_r, \Diamond p_s)$ to ensure that tomorrow is another day: in every possible world intersecting the present, there is a next sunrise followed by a sunset. Otherwise, the second conjunct of Sea$'$ would be witnessed by an intersecting world in which tomorrow never ends, so that $\neg\mathsf{B}(p_s, p_b)$ holds vacuously and the two conjuncts disagree about whether the sun also sets, rather than about whether a sea battle occurs. Given this presupposition, $\mathsf{A}(p_r, \neg\mathsf{B}(p_s, p_b))$ is equivalent to $\neg\mathsf{A}(p_r, \mathsf{B}(p_s, p_b))$, and Sea$'$ asserts that the sea battle's occurrence tomorrow— i.e., $\mathsf{A}(p_r, \mathsf{B}(p_s, p_b))$— is contingent.

[63] Unlike *Determined* from before, Det is refuted by the drift frame $\mathcal{F}^{\circ}$ of **T14** while remaining valid over the DETERMINISTIC translation frame $\mathcal{F}^{1}$, and so the store and recall operators discriminate between $\mathcal{F}^{\circ}$ and $\mathcal{F}^{1}$ where **C4** shows that no sentence without them can.

and worlds $\vec{\mu}$ in a model $\mathcal{M} = \langle W, \mathcal{D}, \Rightarrow, |\cdot|\rangle$ where Sea is true, there are at least two possible worlds $\sigma_1, \sigma_2 \in \langle\tau\rangle_x$ where there is a sea battle in σ_1 at time $y > x$ and there is no sea battle in σ_2 at time y. Considered from the world state $\tau(x) = \sigma_1(x) = \sigma_2(x)$, it would be wrong to claim that σ_1 is the actual world that is destined to be, just as it would be wrong to claim that σ_2 is the actual world. However, when evaluated from $\sigma_1(y)$, there is good reason to take σ_1 to have a clear advantage over σ_2, and *vice versa* when evaluated from $\sigma_2(y)$. After all, a sea battle occurs in σ_1 at time y but not so in σ_2, where this difference cannot be ignored. Even if σ_1 has a better claim to being actual when we arrive at $\sigma_1(y)$, one cannot claim that σ_1 was actual at time x given the intersecting possible world σ_2, and perhaps many others.

The paradox of the open future may be stated as follows: (1) there is a unique actual world; (2) if there is a unique actual world, then there is no future contingency; (3) it appears that there is future contingency. Articulated in terms of the task semantics, the solution is that there is no *actual world* $\alpha \in H_{\mathcal{F}}$ with a determinate past and future. If there were an actual world, one might include it among the semantic primitives in a task frame $\mathcal{F}_{@} = \langle W, \mathcal{D}, \Rightarrow, \alpha\rangle$ where α is taken to at least simulate actuality, providing a semantics for an actuality operator @ as follows:

(@) $\mathcal{M}, \tau, x, \vec{v}, \vec{\mu} \vDash @\varphi$ *iff* $\mathcal{M}, \alpha, x, \vec{v}, \vec{\mu} \vDash \varphi$.

Including an actual world α in a task frame raises questions about the commitments of the intended models for the language. By contrast, $\uparrow_{\text{M}}$ and $\downarrow_{\text{M}}$ store and recall worlds without any reference to a globally actual world, avoiding the apparent commitment to an actual future while still providing expressive resources for cross referencing worlds. Instead of positing an actual world α with a complete past and future, a weaker commitment takes each task frame $\mathcal{F}_{\#} = \langle W, \mathcal{D}, \Rightarrow, \#\rangle$ to include a distinguished *present* $\# \in W$ representing the world state that currently obtains.[64] Given $\#$ and fixing a time n, we may consider all possible worlds that occupy $\#$ at n without any commitment to there being a world that includes an actual future. Although far less committing than $\mathcal{F}_{@}$, the significance of a task frame such as $\mathcal{F}_{\#}$ is fleeting since what obtains now is only momentary. In contrast to both of these proposals, no world state $\# \in W$ has been designated as present, and no possible world $@ \in H_{\mathcal{F}}$ has been designated as the actual world. Instead of positing the structure required to say what is true *simpliciter*, the semantics for $\mathcal{L}$ only includes what is needed to evaluate truth relative to one world and time or another where none are distinguished.

Looking out at the Aegean, Aristotle may have wondered whether there will be a sea battle tomorrow or not. Although there are various possible worlds intersecting with the present according to which there is going to be a sea battle tomorrow as well as possible worlds in which there is not, no possible world is designated as the actual world that is already destined to take place. Rather, it remains open which trajectory through the space of world states we may traverse. By the time tomorrow comes we will know whether there has been a sea battle or not, and so nothing regarding the sea battle will remain open to determination. But arriving where we have by one trajectory or another does not entail that we were always bound to arrive at any given

[64] Alternatively, one might designate a *past* that includes the world states that have obtained up to and including the world state that currently obtains, providing a growing block theory of time with Broad [37].

world state. That we may model our various possible trajectories through the space of world states does not require any trajectory to be actual. At most we may say that the present obtains, though the present is always fleeting.

4.2 Dynamical Systems

Dynamical systems theory provides a general mathematical framework for modeling the possible evolutions of a system, suggesting a natural connection with tense and modal reasoning. Whereas the task frames defined above take time to have both group and order structure, non-deterministic dynamical systems typically consist of a set of states W, a monoid $\langle D, +, 0\rangle$ for positive durations, and a set of relations $\{R_x\}_{x\in D}$ indexed by D that satisfy the following conditions for all $w, u, v \in W$ and $x, y \in D$:[65]

Nullity: $R_0(w, u)$ if and only if $w = u$.

Compositionality: If $R_x(w, u)$ and $R_y(u, v)$, then $R_{x+y}(w, v)$.

Interpolation: If $R_{x+y}(w, v)$, then $R_x(w, u)$ and $R_y(u, v)$ for some $u \in W$.

The relation $R_x(w, u)$ expresses with an indexed family of two-place relations what $w \Rightarrow_x u$ expresses with a three-place relation. Every task frame satisfies the conditions above, where **L9** derives the zero loops of *Nullity*, and *Limit* entails the converse. Since a monoid $\langle D, +, 0\rangle$ supplies only positive durations, composition among them is unrestricted. However, to provide semantic clauses for tense operators, an order is also included where every duration x has an inverse $-x$ and the order of addition does not make a difference. Thus **D19** restricts attention to dynamical systems in which the durations $\mathcal{D} = \langle D, +, 0, \leqslant\rangle$ form a totally ordered abelian group. Given that the task relation induces a canonical topology on world states, **T8** establishes that this topology is *T1* for every task frame, with *R0* following as **C2**.

A dynamical system is *deterministic* if R_x is a total function for all $x \in D$, justifying the notation for the *world functions* $\{f_x\}_{x\in D}$ where $f_x(w) = u$ just in case $R_x(w, u)$. A *deterministic model* is any $\mathcal{M}_D = \langle W, \mathcal{D}, \{f_x\}_{x\in D}, |\cdot|\rangle$ where $|p_i| \subseteq W$ for all $p_i \in \mathbb{L}$ and $\langle W, \mathcal{D}, \{f_x\}_{x\in D}\rangle$ is a deterministic dynamical system. Drawing on these resources, Williamson [4, 5] provides a semantics for a higher-order bimodal language in terms of the primitive world functions provided by a deterministic model. I will present the propositional fragment of Williamson's semantics, adapted to $\mathcal{L}$, as follows:

(p_i) $\mathcal{M}_D, w \vDash p_i$ *iff* $w \in |p_i|$.

($\bot$) $\mathcal{M}_D, w \nvDash \bot$.

($\rightarrow$) $\mathcal{M}_D, w \vDash \varphi \rightarrow \psi$ *iff* $\mathcal{M}_D, w \nvDash \varphi$ or $\mathcal{M}_D, w \vDash \psi$.

($\Box$) $\mathcal{M}_D, w \vDash \Box\varphi$ *iff* $\mathcal{M}_D, u \vDash \varphi$ for all $u \in W$.

($\lhd$) $\mathcal{M}_D, w \vDash \varphi \lhd \psi$ *iff* $\mathcal{M}_D, f_z(w) \vDash \psi$ for some $z < 0$ where $\mathcal{M}_D, f_y(w) \vDash \varphi$ for all $y \in D$ where $z < y < 0$.

[65] A *monoid* is any $\langle M, \cdot, 1_M\rangle$ where: (1) $a \cdot b \in M$ whenever $a, b \in M$; (2) $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ for all $a, b, c \in M$; and (3) $a \cdot 1_M = 1_M \cdot a = a$ for all $a \in M$.

($\rhd$) $\mathcal{M}_D, w \vDash \varphi \rhd \psi$ *iff* $\mathcal{M}_D, f_z(w) \vDash \psi$ for some $z > 0$ where $\mathcal{M}_D, f_y(w) \vDash \varphi$ for all $y \in D$ where $0 < y < z$.

Instead of interpreting sentences at a model, possible world, and time, the semantic clauses above evaluate sentences directly at a model and world state, where the clauses for $\boxed{\scriptstyle P}$ and $\boxed{\scriptstyle F}$ are recovered from those for $\lhd$ and $\rhd$. This approach may claim the advantage of validating the perpetuity principles **P1** − **P6** from before without undermining the intelligibility of the truth-conditions for the sentences of the language. Nevertheless, the semantics above is restricted to deterministic systems, and so unable to model systems in which states fail to determine a unique past and future. For instance, the board states in a game of chess do not typically mandate any particular continuation, leaving open many different incompatible futures.

Since world functions do not accommodate any degree of future contingency, one might attempt to weaken the temporal order in an attempt to encode an open future. As already observed, taking $\mathcal{D}$ to be a partial order presents the difficulty facing the Peircean semantics which quantifies over all or some incomparable future times. For instance, it is absurd to admit that both Black and White are going to win given incomparable future times in which Black and White each win. Instead of attempting to accommodate the open future, it is natural to restrict applications of Williamson's semantics to deterministic systems where exactly one history intersects each world state as **L19** shows. For instance, the motions of the planets are well modeled by a deterministic system constrained under Newton's laws. Given the total state of the solar system including the position and momentum of each planet, all future states of that system are fully determined. If world states determine their past and future, there is no need to specify a temporal parameter alongside a possible world, obviating the need to construct possible worlds. Even so, these are small gains in compensation for the ability to represent non-deterministic systems, especially when metaphysical modality is under discussion. If there are systems whose world states have incompatible futures, the semantics for the strongest objective modality must quantify over all futures for those systems rather than requiring the future to be unique.

The study of dynamical systems is not new. Since the time of Galileo, Newton, and Leibniz during the seventeenth century, it has been standard practice to represent the evolution of a system by using functions. By contrast, it is far from natural not only in physics but throughout the sciences to represent the evolution of a system by a primitive point. The dynamics described by the semantics for languages with tense and modal operators are no different. Thomason [47] presses this as follows:

> Physics should have helped us to realize that a temporal theory of a phenomenon X is, in general, more than a simple combination of two components: the statics of X and the ordered set of temporal instants. The case in which all functions from times to world states are allowed is uninteresting; there are too many such functions, and the theory has not begun until we have begun to restrict them. [...] The general moral, then, is that we shouldn't expect the theory of time + X to be obtained by mechanically combining the theory of time and the theory of X. (p. 135)

Montague [1] and Kaplan's [2] two-dimensional semantics assumes a dynamics that predates the development of functional dynamics during the Scientific Revolution.

Although dynamical systems are nothing new, what has not been adequately explored is the connection between functional dynamics and bimodal reasoning. This paper provides a step in that direction. In the following section, I will conclude by drawing connections to a number of recent developments in logic and computer science.

4.3 Conclusion

The construction of possible worlds validates the perpetuity principles **P1** – **P6** while maintaining an expressive semantics for tense and modality. These principles are derived from the bimodal axiom **MF**, which records in the object language the free translation of possible worlds that the construction secures. Indeterminacy is located in the task relation, which may relate a given world state to more than one world state over a fixed duration, rather than by the incomparable times posited by those following Prior. The possible worlds intersecting a given world state may each realize a different way for the system to evolve, and so the task semantics admits an open future without weakening the temporal order. The soundness and completeness of **TM** and its extensions, verified in the Lean 4 repository for this paper and sketched in **T18** and **C6**, demonstrate that the task semantics is both correct and adequate for reasoning about non-deterministic dynamical systems.

These results address the deficiencies in extant bimodal logics. Whereas Montague [1] works with a less expressive modality that trivializes the perpetuity principles, Kaplan's [2] semantics invalidates perpetuity. Restricting to the abundant models in §**2.3** restores validity, but only by imposing a model constraint that undermines the truth-conditions for the language. Since *Abundance* is not preserved under a change of valuation, abundance theorists must posit all merely temporal differences between primitive worlds, choosing between either temporal absolutism or instrumentalism. By contrast, Williamson's [4] semantics validates the perpetuity principles, but restricts attention to systems in which each world state fixes a unique future, and so cannot accommodate non-deterministic dynamical systems. By introducing a task relation to constrain transitions between world states over a duration without requiring the task relation to be functional, the task semantics handles non-determinism, validates the perpetuity principles, and avoids temporal absolutism, all while preserving the expressive power of the semantic clauses for the tense and modal operators.

The task semantics is closely related to several neighboring research programs. Belnap, Perloff, and Xu's [48] agentive `STIT` theory and Rumberg's [28] transition semantics follow Thomason's [26] reconstruction of Prior's [22] Ockhamist semantics. Both frameworks represent possibilities without committing to a determined actual future by taking times to be partially ordered $\langle T, \prec \rangle$, admitting incomparable times that branch into alternative futures. In opposition to this approach, the task semantics defines the models of $\mathcal{L}$ over a task frame $\mathcal{F} = \langle W, \mathcal{D}, \Rightarrow \rangle$ where a task $w \Rightarrow_x u$ encodes the transition between world states $w, u \in W$ over the duration $x \in D$ and $\mathcal{D} = \langle D, +, 0, \leqslant \rangle$ is a totally ordered abelian group. The possible worlds are the labeled runs of $\mathcal{F}$, where the branching tree of states is not posited but may be recovered by unfolding the transitions that the task relation permits. Since the domain of each possible world is totally ordered by $\mathcal{D}$, the tense operators quantify over the *times*— i.e., the durations after 0— in a single world while the modal operators quantify over alternative worlds, avoiding the Peircean difficulties of **§2.1** that arise from doing both in one clause. The agentive choice function that Belnap, Perloff, and Xu add to branching time, partitioning the histories through a moment into an agent's available choices, could likewise be overlaid on the possible worlds intersecting at a world state and time to define agentive operators over the dynamics developed here.[66] The first-order definability and axiomatizability results that Rumberg and Zanardo [50] establish provide a useful benchmark that the present framework meets: **T10**, **T11**, and **T12** provide the corresponding definability results, where the completeness result **C6** for the $\mathcal{L}$ systems extends the benchmark to the discrete, dense, and continuous classes.[67] Rather than ensuring axiomatizability by adding a freely chosen bundle of histories as a further component of the models, $H_{\mathcal{F}}$ is the maximal set of possible worlds determined by a task frame $\mathcal{F}$.

The debate in computer science between branching-time and linear-time temporal logics receives a distinctive resolution in the present framework.[68] Individual possible worlds are linearly ordered as in LTL [55], while the set of possible worlds through a given world state has a branching structure as in CTL [56]. The key difference is that $\Box$ quantifies universally over all possible worlds in $H_{\mathcal{F}}$ at the same time rather than employing the state-relativized path quantifiers A and E of CTL. Together with the closure of $H_{\mathcal{F}}$ under translation, this universal quantification validates **MF** and hence the perpetuity principles, as shown in **§3.2**. Whereas **TM** validates its bimodal interaction principle by the construction of possible worlds itself, this fails in the broader landscape of products of modal logics.[69] We may also restrict quantification to intersecting worlds with $\boxdot$, or to worlds with a common past and present relative to the world and time of evaluation by adding $\boxed{\triangleright}$. HyperLTL provides a parallel treatment by extending LTL with quantifiers $\forall\pi$ and $\exists\pi$ ranging over the execution traces of a system. Rather than a bimodal logic with tense and modal operators, HyperLTL is a tense logic with first-order quantifiers for traces, making satisfiability in HyperLTL undecidable.[70] By contrast, the semantics for $\Box$ quantifies over the possible worlds in $H_{\mathcal{F}}$ without binding traces within the object language, where the Lean 4 repository for this paper implements a decision procedure for **TM** whose soundness is verified, though no decidability theorem for **TM** is machine-checked at present.[71]

[66] See Horty [49] for a `STIT`-based treatment of agentive operators.

[67] Definability and axiomatizability come apart for the DETERMINISTIC task frames. Although no set of sentences of $\mathcal{L}$ extended with $\boxdot$ defines them by **C4**, their logic in that language is completely axiomatized by **TM** together with *Determined* and principles governing $\boxdot$. Definability is recovered once the store and recall operators are added. Both results are established in the Lean 4 repository for this paper.

[68] The debate was initiated by Lamport [51] and later given its "final showdown" by Vardi [52]. Emerson and Halpern's CTL* [53] subsumes both LTL and CTL by combining path quantifiers freely with temporal operators. Lamport's TLA [54] takes as primitive a relation between old and new states that, like the task relation, need not be functional. However, TLA's $\Box$ is a tense operator, ranging over the suffixes of one *behavior*— i.e., an ω-sequence of states— where no operator quantifies over the alternative continuations.

[69] See Kurucz [32] for a general treatment of the products of modal logics. The task relation is also structurally analogous to the program relation in the dynamic logic that Harel, Kozen, and Tiuryn [57] build on Pratt's [58] relational treatment of programs, though the labels in the present framework carry abelian group structure that dynamic logic does not require. Its propositional fragment is moreover EXPTIME-complete by Fischer and Ladner's [59] closure construction, unlike the still-open decidability of **TM**.

[70] Clarkson et al. [60] introduced HyperLTL, and Finkbeiner and Zimmermann [61] showed that it translates into a proper fragment of first-order logic with an equal-level predicate relating positions at the same time on distinct traces. Finkbeiner and Hahn [62] proved that its satisfiability problem is undecidable, whereas Finkbeiner, Rabe, and Sánchez [63] showed that model checking against a given finite-state system is decidable, with complexity rising by an exponential for each quantifier alternation.

[71] See the repository for the ongoing effort to verify decidability.

5 Appendix

This section presents the formal results supporting the main body of the paper. In §5.1, I characterize $\Box$ as the strongest objective modal operator, proving it unique up to equivalence and deriving **4** from composition, **B** from converse and reversal, and **T** from these together with the consistency of the strongest objective modality. An axiomatic theory of the objective modalities then shows that every objective modality is normal in **L3** and establishes existence in **T1** without assuming Booleanism.

After §5.2 presents the results supporting the arguments against two-dimensional semantics in §2, the task semantics for the language $\mathcal{L}$ is restated in §5.3, establishing the frame correspondence and determinism results together with the topology that a task frame induces on its world states. §5.4 sketches the soundness of **TM** and presents its extensions, where **C6** establishes their completeness. The results of §5.3 and §5.4 are formalized in the Lean 4 repository for this paper.[72]

5.1 Objective Modality

This section characterizes metaphysical modality as the *strongest objective modality* where objective modality is axiomatized by a predicate O on operators. An auxiliary operator constant B that satisfies this characterization establishes existence in **T1** before showing that $\Box$ is equivalent to B in **C1**. The treatment is predicative where operator comprehension is confined to formulas containing no operator variables and no occurrences of O. This keeps the system consistent with a fine-grained theory of propositional identity.[73] Since tense is not at issue, I will consider a purely modal language fragment that is extended by the propositional identity operator and higher-order quantifiers binding variables in both sentence and unary operator propositions.

D1 Letting $\mathcal{L}^{\Box}$ be the purely modal fragment of $\mathcal{L}$, each $p_i \in \mathbb{L}$ will be understood to be a *propositional variable* rather than sentence letter. The *identity extension* of $\mathcal{L}^{\Box}$ is a language $\mathcal{L}^{\equiv}$ enriched to include a binary propositional identity operator $\equiv$, read ⌜For φ just is for ψ⌝, whose logic comprises classical propositional logic and the minimal theory of identity given below, where $\chi_{(\psi/\varphi)}$ is the result of replacing one or more occurrences of φ in any formula χ with ψ:

Ref $\vdash \varphi \equiv \varphi$.

Imp $\vdash (\varphi \equiv \psi) \to (\varphi \to \psi)$.

LL$^-$ $\vdash (\varphi \equiv \psi) \to (\chi \to \chi_{(\psi/\varphi)})$ where ψ is free for φ in χ and no replaced φ lies in the scope of an operator.[74]

The theory of propositional identity need not be Boolean, accommodating theories in which the absorption laws or other Boolean identities do not hold.[75] Symmetry and

[72] See the repository for this paper.

[73] Predicativity is a standard response to Russell–Myhill. See Walsh [64] for an explicit consistency proof in Church's intensional logic. Predicativity could be dropped by adopting Bacon's [6] *Rule of Equivalence*—that interderivable formulas are identical—to coarsen $\equiv$ enough to block Russell–Myhill on its own. Bacon [6, p. 747] shows that the rule is strictly stronger than *Booleanism*— that Boolean-equivalent formulas are identical— which I will not assume here to maintain greater generality.

[74] A formula ψ is *free for* φ in χ just in case no replaced occurrence of φ lies within the scope of a quantifier binding a variable free in φ or ψ. The condition is vacuous in $\mathcal{L}^{\equiv}$, which has no quantifiers.

[75] I defend a bilateral theory of propositional identity in Brast-McKie [65].

transitivity of $\equiv$ are nevertheless derivable, and since each replaces an occurrence lying outside any operator term, both survive the proviso on **LL**$^-$.

D2 The *quantified identity extension* $\mathcal{L}^{\equiv}_{\forall}$ of $\mathcal{L}^{\equiv}$ adds a countable stock of unary operator variables $\mathcal{Q}, \mathcal{P}, \ldots$ with operator quantifiers $\forall\mathcal{Q}$ to bind them, along with propositional quantifiers $\forall p$ binding the propositional variables. The *operator terms* are generated from the operator variables, the modal primitive $\Box$, and a primitive operator constant B closed under composition $\circ$ and converse $\smile$:[76]

$$T,\ T' \ ::=\ \mathcal{Q} \mid \Box \mid \mathsf{B} \mid T \circ T' \mid T^{\smile},$$

and the *well-formed formulas* are defined recursively:

$$\varphi,\ \psi\ ::=\ p_i \mid \bot \mid \varphi \to \psi \mid \varphi \equiv \psi \mid T\varphi \mid \mathsf{O}(T) \mid \forall p\, \varphi \mid \forall \mathcal{Q}\, \varphi.$$

The operator quantifier $\forall\mathcal{Q}$ binds one-place operator variables. The formula $\mathsf{O}(T)$ reads ⌜T is an objective modality⌝, where the predicate O and the B are governed by the postulates below. The dual of an operator T is defined as $\langle T\rangle\varphi \coloneqq \neg T\neg\varphi$.

R1 The constant B is an auxiliary, and it is the one operator whose objectivity is postulated outright: **O-Meet** stipulates that B is objective and that B requires just what every objective modality requires, so that a strongest objective modality can be shown to exist without presupposing which operator it is. By contrast, the modal primitive $\Box$ is not constrained by axioms, and the converse term-formation operator $\smile$ only by a single law for objective modalities. Since $\Box$ is the operator under study, leaving it unconstrained is what makes the hypothesis that it is strongest among the objective modalities a substantive claim about metaphysical necessity rather than a syntactic stipulation. The predicate O is also primitive rather than defined, since it is in terms of O that both the auxiliary constant and the hypothesis about $\Box$ are stated.

D3 A *logic L for $\mathcal{L}^{\equiv}_{\forall}$* extends the logic from **D1** to include the axioms and rules below, where $\varphi[\psi/p]$ replaces every free occurrence of p by ψ with ψ free for p, and $\varphi[T/\mathcal{Q}]$ replaces every free occurrence of $\mathcal{Q}$ by the operator term T with T free for $\mathcal{Q}$:

- **I1** $\vdash \forall p\, \varphi \to \varphi[\psi/p]$
- **I2** $\vdash (T \circ T')\varphi \equiv T(T'\varphi)$
- **I3** $\vdash \forall \mathcal{Q}\, \varphi \to \varphi[T/\mathcal{Q}]$
- **I4** $\vdash \varphi$ implies both $\vdash \forall p\, \varphi$ and $\vdash \forall\mathcal{Q}\, \varphi$
- **I5** $\vdash \forall p(\varphi \to \psi) \to (\varphi \to \forall p\, \psi)$ provided the variable p is not free in φ
- **I6** $\vdash \forall \mathcal{Q}(\varphi \to \psi) \to (\varphi \to \forall \mathcal{Q}\, \psi)$ provided the variable $\mathcal{Q}$ is not free in φ
- **I7** $\vdash \exists \mathcal{Q}\, \forall p\, (\mathcal{Q}p \equiv \theta)$ for θ containing no operator variables, no occurrences of O, and at most p free (where $\exists\mathcal{Q}$ abbreviates $\neg\forall\mathcal{Q}\neg$).

[76] B must enter as a primitive constant rather than a defined operator term: its defining condition $\forall\mathcal{P}(\mathsf{O}(\mathcal{P}) \to \mathcal{P}p)$ contains both an operator variable and an occurrence of O, each barred from predicative comprehension, so no comprehension instance can express it.

D4 Objective modality is characterized with the following operator predicates:

Fac $\mathrm{Fac}(\mathcal{Q}) \coloneqq \forall p(\mathcal{Q}p \to p)$.

Ax $\mathrm{Ax}(\mathcal{Q}) \coloneqq \mathcal{Q}\top \wedge \forall p \forall q[\mathcal{Q}(p \to q) \to (\mathcal{Q}p \to \mathcal{Q}q)]$, where $\top \coloneqq (\bot \to \bot)$.

Ord $\mathcal{Q} \leq \mathcal{P} \coloneqq \forall p(\mathcal{Q}p \to \mathcal{P}p)$, read ⌜$\mathcal{Q}$ is at least as strong as $\mathcal{P}$⌝.

Briefly, **Fac** states factivity, **Ax** collects the modal axioms **N** and **K**, and **Ord** orders operators by relative strength.

D5 A logic L for $\mathcal{L}^{\equiv}_{\forall}$ is a *theory of objective modalities* just in case it includes:

O-Ax $\vdash \forall \mathcal{Q}(\mathsf{O}(\mathcal{Q}) \to \mathrm{Ax}(\mathcal{Q}))$.

O-Nec $\vdash \varphi$ implies $\vdash \forall \mathcal{Q}(\mathsf{O}(\mathcal{Q}) \to \mathcal{Q}\varphi)$.

O-Tran $\vdash \forall \mathcal{Q}[\mathsf{O}(\mathcal{Q}) \to \forall p \forall q((p \equiv q) \to (\mathcal{Q}p \equiv \mathcal{Q}q))]$.

O-Comp $\vdash \forall \mathcal{Q} \forall \mathcal{P}[(\mathsf{O}(\mathcal{Q}) \wedge \mathsf{O}(\mathcal{P})) \to \mathsf{O}(\mathcal{Q} \circ \mathcal{P})]$.

O-Conv $\vdash \forall \mathcal{Q}(\mathsf{O}(\mathcal{Q}) \to \mathsf{O}(\mathcal{Q}^{\smile}))$.

O-Meet $\vdash \forall p\,(\mathsf{B}p \leftrightarrow \forall \mathcal{P}(\mathsf{O}(\mathcal{P}) \to \mathcal{P}p))$ and $\vdash \mathsf{O}(\mathsf{B})$.

O-Rev $\vdash \forall \mathcal{Q}[\mathsf{O}(\mathcal{Q}) \to \forall p(p \to \mathcal{Q}\langle \mathcal{Q}^{\smile} \rangle p)]$.[77]

O-Cons $\vdash \neg \mathsf{B}\bot$.

The postulates above are proven consistent in **P1** and divide into groups: normality (**O-Ax**, **O-Nec**), transparency (**O-Tran**), closure (**O-Comp**, **O-Conv**, **O-Meet**), reversal (**O-Rev**), and consistency (**O-Cons**), where **T2** derives **4** from composition, **B** from converse and reversal, and **T** from these together with the consistency of the strongest objective modality. **O-Tran** is the unrestricted unary form of *Transparency* from **§1**, which is substantive since **LL**$^-$ does not reach inside operator terms to avoid restricting the language to transparent operators. Whereas **O-Ax** requires every objective modality to satisfy **N** and **K**, the rule **O-Nec** closes L under necessitation for all objective operators.[78] The closure postulates close the objective modalities under composition, converse, and their infinitary conjunction, where **O-Meet** asserts that B requires just what every objective modality requires and is itself objective.[79] **O-Cons** says that B is consistent and is stated for B alone since the class-wide claim $\forall \mathcal{Q}(\mathsf{O}(\mathcal{Q}) \to \neg \mathcal{Q}\bot)$ is strictly stronger and is not among the postulates.

[77] The restriction to O operators is forced: comprehension **I7** at $\theta \coloneqq \bot$ supplies a $\mathcal{Q}$ with $\forall p(\mathcal{Q}p \equiv \bot)$, and unguarded **O-Rev** at this $\mathcal{Q}$ gives $p \to \mathcal{Q}\langle \mathcal{Q}^{\smile} \rangle p$, whose consequent yields $\bot$ by **Imp**, detaching $\vdash \neg p$ for every p and so contradicting $\vdash \top$. Converse can be governed only on a class excluding such operators.

[78] Here **N** names the axiom $\mathcal{Q}\top$, as in Chellas [66, §4.1], not the rule of necessitation. The two come apart because $\vdash \varphi$ does not yield $\vdash \varphi \equiv \top$ given the minimal theory of propositional identity given above, and so **N** and **K** do not by themselves close L under $\mathcal{Q}$-necessitation, so **O-Nec** is needed as a separate rule.

[79] **O-Nec** and **O-Meet** correspond to *Necessitation* and *L-Necessity* in Bacon and Zeng's theory of necessities [67], where Thm. 3.1 is their existence result. **O-Comp** corresponds to their *Composition*, which is a theorem [67, Prop. 2.3] rather than a postulate. The **T** axiom is derived here in **T2** from the consistency of the strongest objective modality together with closure and reversal, not from an identity-membership postulate, marking a point of divergence from their derivation of *Identity* as a theorem [67, Prop. 2.2].

L1 *If* $\vdash \mathsf{O}(T)$*, then* $\vdash \forall p \forall q\,[(p \equiv q) \to (Tp \equiv Tq)]$.

Proof. Instantiating **O-Tran** at T by **I3** and detaching given the assumption $\vdash \mathsf{O}(T)$ yields $\vdash \forall p \forall q\,[(p \equiv q) \to (Tp \equiv Tq)]$ as desired. $\square$

D6 Let $\mathcal{W}$ be a nonempty set of possible worlds and $\mathcal{C}$ a class of relations on $\mathcal{W}$.[80] A *relational interpretation induced by* $\mathcal{C}$ is given by the following:

Successor: For worlds $\sigma, \tau \in \mathcal{W}$, σ is an *R-successor* of τ just in case $\langle \tau, \sigma \rangle \in R$.

Converse: $R^{\smile} := \{\langle \sigma, \tau \rangle \mid \langle \tau, \sigma \rangle \in R\}$.

Relative Product: $R\,;S := \{\langle \tau, \rho \rangle \mid \langle \tau, \sigma \rangle \in R$ and $\langle \sigma, \rho \rangle \in S$ for some $\sigma\}$.

Identity: $Id_{\mathcal{W}} := \{\langle \tau, \tau \rangle \mid \tau \in \mathcal{W}\}$.

Propositions: Propositional variables range over $\wp(\mathcal{W})$, where a formula is true at τ just in case τ belongs to the proposition it expresses, with $\bot$ and $\to$ interpreted classically and $\varphi \equiv \psi$ true just in case φ and ψ express the same proposition.

World-quantifiers: For each relation R on $\mathcal{W}$, the operation $\mathcal{Q}_R$ takes a proposition X to the set of worlds $\mathcal{Q}_R(X) := \{\tau \in \mathcal{W} \mid \sigma \in X$ for every σ with $\langle \tau, \sigma \rangle \in R\}$, where distinct relations determine distinct world-quantifiers.[81]

Operations: Operator variables range over all operations on $\wp(\mathcal{W})$, where $\circ$ denotes composition, $\square$ an arbitrary operation, B the world-quantifier $\mathcal{Q}_{\bigcup \mathcal{C}}$ of the union of $\mathcal{C}$, and $\smile$ any operation on operations taking $\mathcal{Q}_R$ to $\mathcal{Q}_{R^{\smile}}$ for every R on $\mathcal{W}$.[82]

Objectivity: $\mathsf{O}(T)$ expresses $\mathcal{W}$ if T denotes $\mathcal{Q}_R$ for some $R \in \mathcal{C}$, and $\varnothing$ otherwise.

L2 *Every relational interpretation validates the axioms and rules of **D1**, **D3**, **O-Ax**, **O-Nec**, **O-Rev**, and **O-Tran**, whatever* $\mathcal{C}$ *may be, and moreover:*

1. $\exists \mathcal{Q}(\mathsf{O}(\mathcal{Q}) \wedge \forall p(\mathcal{Q}p \equiv p))$ *is valid just in case* $Id_{\mathcal{W}} \in \mathcal{C}$.
2. ***O-Comp*** *is valid if* $\mathcal{C}$ *is closed under composition.*
3. ***O-Conv*** *is valid if* $\mathcal{C}$ *is closed under converse.*
4. ***O-Meet*** *is valid if* $\bigcup \mathcal{C} \in \mathcal{C}$.
5. $\forall \mathcal{Q}(\mathsf{O}(\mathcal{Q}) \to \mathrm{Fac}(\mathcal{Q}))$ *is valid if every* $R \in \mathcal{C}$ *is reflexive.*
6. ***O-Cons*** *is valid just in case* $\bigcup \mathcal{C}$ *is serial.*

[80] Since $\mathcal{L}^{\equiv}_{\forall}$ extends the tense-free fragment $\mathcal{L}^{\square}$, no operator shifts the time coordinate, which may accordingly be suppressed; taking $\mathcal{W} := H_{\mathcal{F}}$ recovers the task semantics.

[81] Each relation is recoverable from its world-quantifier, since $\tau \in \mathcal{Q}_R(\mathcal{W} \backslash \{\sigma\})$ just in case $\langle \tau, \sigma \rangle \notin R$, and so $\mathcal{Q}_S = \mathcal{Q}_R$ only if $S = R$. The interpretation of $\smile$ below is thereby well defined.

[82] The interpretation must be total since **I3** instantiates operator quantifiers at $T^{\smile}$ for any operator term T, though $\smile$ may act arbitrarily on the operations other than the world-quantifiers, since **O-Conv** and **O-Rev** apply only to operators satisfying O. Some such operation exists, since any total extension of the function $\mathcal{Q}_R \mapsto \mathcal{Q}_{R^{\smile}}$ will suffice.

Proof. Fix a relational interpretation induced by $\mathcal{C}$ as in **D6**. The logic of **D1** is valid: classical propositional logic because $\bot$ and $\to$ are interpreted classically, and **Ref**, **Imp**, and **LL$^-$** because $\equiv$ is identity of propositions and the interpretation is compositional. The axioms and rules of **D3** are valid: **I1** and **I3** because instantiation stays within the range of the quantifiers, **I2** because $\circ$ is composition, **I4** because validity is truth under every assignment, **I5** and **I6** because the value of φ does not depend on a variable not free in it, and **I7** because θ, having no operator variables and at most p free, expresses an operation on propositions.

Of the postulates, **O-Tran** is valid outright, since operator variables range over operations on $\wp(\mathcal{W})$, so co-denoting arguments yield co-denoting values whatever $\mathcal{Q}$ denotes. The rest are guarded by O, and since $\mathsf{O}(\mathcal{Q})$ is true only when $\mathcal{Q}$ denotes $\mathcal{Q}_R$ for some $R \in \mathcal{C}$, we may assume that $\mathcal{Q}$ denotes $\mathcal{Q}_R$ and $\mathcal{P}$ denotes $\mathcal{Q}_S$ for some $R, S \in \mathcal{C}$. Then $\mathcal{Q}\varphi$ is true at τ just in case φ is true at every R-successor of τ, $\mathcal{Q}^{\smile}$ denotes $\mathcal{Q}_{R^{\smile}}$, and $\mathcal{Q} \circ \mathcal{P}$ denotes $\mathcal{Q}_R \circ \mathcal{Q}_S = \mathcal{Q}_{R;S}$. For **O-Ax**, $\mathcal{Q}\top$ is true at every world since $\top$ is also, and $\mathcal{Q}(p \to q) \to (\mathcal{Q}p \to \mathcal{Q}q)$ since an R-successor at which $p \to q$ and p are true is one at which q is true. For **O-Nec**, if φ is valid, then φ is true at every R-successor of every world, so $\mathcal{Q}\varphi$ is valid. For **O-Rev**, suppose that p is true at τ and consider any R-successor σ of τ: since $\langle \tau, \sigma \rangle \in R$, the world τ is an $R^{\smile}$-successor of σ at which p is true, so $\langle \mathcal{Q}^{\smile} \rangle p$ is true at σ, and since σ was an arbitrary R-successor, it follows that $\mathcal{Q}\langle \mathcal{Q}^{\smile} \rangle p$ is true at τ.

For (1), the identity operation is $\mathcal{Q}_{Id_{\mathcal{W}}}$, since $\mathcal{Q}_{Id_{\mathcal{W}}}(X) = X$ for every proposition X, and it is $\mathcal{Q}_R$ for no other R, so some $\mathcal{Q}$ with $\mathsf{O}(\mathcal{Q})$ satisfies $\forall p(\mathcal{Q}p \equiv p)$ just in case $Id_{\mathcal{W}} \in \mathcal{C}$. For (2) and (3), given that $\mathsf{O}(\mathcal{Q})$ and $\mathsf{O}(\mathcal{P})$ are true, $\mathcal{Q} \circ \mathcal{P}$ denotes $\mathcal{Q}_{R;S}$ and $\mathcal{Q}^{\smile}$ denotes $\mathcal{Q}_{R^{\smile}}$ with $R, S \in \mathcal{C}$, so $\mathsf{O}(\mathcal{Q} \circ \mathcal{P})$ is true just in case $R\,;S \in \mathcal{C}$ and $\mathsf{O}(\mathcal{Q}^{\smile})$ just in case $R^{\smile} \in \mathcal{C}$. Closure of $\mathcal{C}$ under composition and converse therefore validates **O-Comp** and **O-Conv**. For (4), let p express X. Since a $\bigcup\mathcal{C}$-successor of τ is just an R-successor of τ for some $R \in \mathcal{C}$, we have $\mathcal{Q}_{\bigcup\mathcal{C}}(X) = \bigcap_{R \in \mathcal{C}} \mathcal{Q}_R(X)$. As B denotes $\mathcal{Q}_{\bigcup\mathcal{C}}$, $\mathsf{B}p$ is true at τ just in case $\tau \in \mathcal{Q}_{\bigcup\mathcal{C}}(X)$. Given that $\mathsf{O}(\mathcal{P})$ is true just in case $\mathcal{P}$ denotes $\mathcal{Q}_R$ for some $R \in \mathcal{C}$, we get $\forall\mathcal{P}(\mathsf{O}(\mathcal{P}) \to \mathcal{P}p)$ is true at τ just in case $\tau \in \mathcal{Q}_R(X)$ for every $R \in \mathcal{C}$. By the identity above these coincide, so the first conjunct of **O-Meet** is valid, while the second, $\mathsf{O}(\mathsf{B})$, is true just in case $\bigcup\mathcal{C} \in \mathcal{C}$. For (5), if R is reflexive then τ is among its own R-successors, so $\mathcal{Q}p \to p$ is valid. For (6), B denotes $\mathcal{Q}_{\bigcup\mathcal{C}}$, and $\mathsf{B}\bot$ is true at τ just in case τ has no $\bigcup\mathcal{C}$-successor, so $\neg\mathsf{B}\bot$ is valid just in case every world has a successor, that is, just in case $\bigcup\mathcal{C}$ is serial. $\square$

P1 *The theory of objective modalities given in **D5** is consistent.*

Proof. Take any nonempty set $\mathcal{W}$ and consider the relational interpretation it induces in **D6** where $\mathcal{C}$ is the class of all relations $R \subseteq \mathcal{W} \times \mathcal{W}$. The class is closed under composition and converse, validating **O-Comp** and **O-Conv** by clauses (2) and (3) of **L2**. The class's union $\bigcup\mathcal{C} = \mathcal{W} \times \mathcal{W}$ belongs to the class, validating **O-Meet** by clause (4). Since $\mathcal{W}$ is nonempty, $\mathcal{W} \times \mathcal{W}$ is serial, validating **O-Cons** by clause (6). Here B denotes $\mathcal{Q}_{\mathcal{W}\times\mathcal{W}}$, where we may interpret $\Box$ as that same operation so that the least objective modality is the unrestricted quantifier, as maintained in **§3.1**. Every relational interpretation validates the remaining axioms and postulates included in **L2**, and so everything in **D3** and **D5** is thereby validated.

Whereas **O-Meet** asserts the existence of the meet of all objective modal operators, this model validates the stronger claim that every subclass has a meet in the class, since $\bigcap_i \mathcal{Q}_{R_i}(X) = \mathcal{Q}_{\bigcup_i R_i}(X)$ for every proposition X and a union of relations on $\mathcal{W}$ is again in the class.[83] $\square$

P2 ***O-Conv*** *is not derivable from the remaining postulates of* ***D5****, and neither is the* ***B*** *axiom nor the* ***T*** *axiom for* B*, even in the presence of the* *4* *axiom.*

Proof. Let $\mathcal{W} \coloneqq \{a, b\}$, $R_0 \coloneqq \{\langle a, b\rangle, \langle b, b\rangle\}$, and $\mathcal{C} \coloneqq \{R_0\}$ given the interpretation in **D6**. The class is closed under composition since $R_0 \mathbin{;} R_0 = R_0$, contains its union since $\bigcup \mathcal{C} = R_0$, and consists of a serial relation. By clauses (2), (4), and (6) of **L2**, together with the validity of **O-Ax**, **O-Nec**, **O-Rev**, and **O-Tran** in every relational interpretation, every postulate but **O-Conv** holds, with B denoting $\mathcal{Q}_{R_0}$.

Yet **O-Conv** fails at $\mathcal{Q} \coloneqq \mathcal{Q}_{R_0}$, since $R_0^{\smile} = \{\langle b, a\rangle, \langle b, b\rangle\} \notin \mathcal{C}$. For p true at b alone, $\mathsf{B}p$ holds at a, whose only R_0-successor is b, while p is false at a, so the **T** axiom fails. For p true at a alone, $\neg p$ holds at b, the only R_0-successor of both a and b, so $\mathsf{B}\neg p$ holds at both worlds, making $\langle \mathsf{B}\rangle p \coloneqq \neg\mathsf{B}\neg p$ fail at both; since a's only successor b fails $\langle \mathsf{B}\rangle p$, $\mathsf{B}\langle \mathsf{B}\rangle p$ fails at a, where p holds, so the **B** axiom fails. Since R_0 is transitive, the **4** axiom holds, so **B** and **T** fail even given **4**. $\square$

D7 $\mathcal{Q}$ is a *normal modal operator in a logic L*— $\mathrm{Norm}_L(\mathcal{Q})$— if and only if:

1. $\vdash \mathrm{Ax}(\mathcal{Q})$; and
2. L is closed under $\mathcal{Q}$-necessitation: $\vdash \varphi$ implies $\vdash \mathcal{Q}\varphi$.

L3 *Any operator term T with $\vdash \mathsf{O}(T)$ is normal in L, and hence monotone so that $\vdash \alpha \to \beta$ implies $\vdash T\alpha \to T\beta$.*

Proof. We get $\vdash \mathrm{Ax}(T)$ by instantiating **O-Ax** at T and detaching. Additionally, L is closed under T-necessitation since **O-Nec** instantiates to $\vdash \mathsf{O}(T) \to T\varphi$, and so $\vdash T\varphi$ again by detachment. Given $\vdash \alpha \to \beta$, T-necessitation yields $\vdash T(\alpha \to \beta)$, and so the **K** conjunct of $\mathrm{Ax}(T)$ yields $\vdash T\alpha \to T\beta$ by detachment. $\square$

D8 $\mathcal{Q}$ is a *strongest objective modal operator in L*— $\mathrm{Str}^{\mathsf{O}}_L(\mathcal{Q})$— if and only if:

1. $\vdash \mathsf{O}(\mathcal{Q})$; and
2. $\vdash \forall \mathcal{P}[\mathsf{O}(\mathcal{P}) \to (\mathcal{Q} \leq \mathcal{P})]$.

T1 $\mathrm{Str}^{\mathsf{O}}_L(\mathsf{B})$*, so L includes a strongest objective modal operator.*

[83] Narrowing $\mathcal{C}$ to the *reflexive* relations preserves every postulate, since reflexivity survives composition, converse, and union, and the union remains the total relation, while also making every objective modality factive by clause (5) of **L2**, a property the class as given lacks since it contains the empty relation. The theory does not assume class-wide factivity to maintain greater generality.

Proof. Clause (1) is the second conjunct of **O-Meet**. For clause (2), the first conjunct of **O-Meet** yields $\vdash \mathsf{B}p \to (\mathsf{O}(\mathcal{P}) \to \mathcal{P}p)$ by instantiating at p and $\mathcal{P}$, so permuting antecedents and generalizing gives $\vdash \forall\mathcal{P}[\mathsf{O}(\mathcal{P}) \to (\mathsf{B} \preceq \mathcal{P})]$. □

T2 *If* $\mathrm{Str}^{\mathsf{O}}_{L}(\mathcal{Q})$, *then* $\vdash \forall p(\mathcal{Q}p \to \mathcal{Q}\mathcal{Q}p)$, $\vdash \forall p(p \to \mathcal{Q}\langle\mathcal{Q}\rangle p)$, *and* $\vdash \forall p(\mathcal{Q}p \to p)$.

Proof. We treat the three axioms in order of dependence— **4**, **B**, **T**.

For (**4**), instantiating **O-Comp** and detaching gives $\vdash \mathsf{O}(\mathcal{Q} \circ \mathcal{Q})$, so dominance yields $\vdash \mathcal{Q}p \to (\mathcal{Q} \circ \mathcal{Q})p$. Since $\vdash (\mathcal{Q} \circ \mathcal{Q})p \equiv \mathcal{Q}\mathcal{Q}p$ by **I2** and $\vdash (\mathcal{Q} \circ \mathcal{Q})p \to \mathcal{Q}\mathcal{Q}p$ by **Imp**, we know $\vdash \forall p(\mathcal{Q}p \to \mathcal{Q}\mathcal{Q}p)$ follows by chaining and generalizing.

For (**B**), instantiating **O-Conv** and detaching gives $\vdash \mathsf{O}(\mathcal{Q}^{\smile})$, so dominance yields $\vdash \mathcal{Q}\neg p \to \mathcal{Q}^{\smile}\neg p$. Contraposing yields $\vdash \langle\mathcal{Q}^{\smile}\rangle p \to \langle\mathcal{Q}\rangle p$, which $\mathcal{Q}$-monotonicity lifts to $\vdash \mathcal{Q}\langle\mathcal{Q}^{\smile}\rangle p \to \mathcal{Q}\langle\mathcal{Q}\rangle p$. Instantiating **O-Rev** and detaching gives $\vdash p \to \mathcal{Q}\langle\mathcal{Q}^{\smile}\rangle p$, and so $\vdash \forall p(p \to \mathcal{Q}\langle\mathcal{Q}\rangle p)$ by chaining and generalizing.

For (**T**), $\vdash \mathsf{O}(\mathsf{B})$ by **O-Meet**, and so $\vdash \mathcal{Q}\bot \to \mathsf{B}\bot$ by dominance. Contraposing and detaching with **O-Cons** yields $\vdash \neg\mathcal{Q}\bot$. Since $\vdash \varphi \to (\neg\varphi \to \bot)$ classically, normality gives $\vdash \mathcal{Q}\varphi \to (\mathcal{Q}\neg\varphi \to \mathcal{Q}\bot)$, and so $\vdash \mathcal{Q}\varphi \to \neg\mathcal{Q}\neg\varphi$ by $\vdash \neg\mathcal{Q}\bot$ and classical logic, which is the **D** axiom $\vdash \mathcal{Q}\varphi \to \langle\mathcal{Q}\rangle\varphi$. By then instantiating at $\mathcal{Q}p$, we get $\vdash \mathcal{Q}\mathcal{Q}p \to \langle\mathcal{Q}\rangle\mathcal{Q}p$, where the dual of **B** gives $\vdash \langle\mathcal{Q}\rangle\mathcal{Q}p \to p$. Instantiating **4** gives $\vdash \mathcal{Q}p \to \mathcal{Q}\mathcal{Q}p$, and so $\vdash \forall p(\mathcal{Q}p \to p)$ by chaining and generalizing. □

R2 The proof of **T2** appeals to **O-Cons** only to detach $\neg\mathcal{Q}\bot$, and **T1** gets $\mathrm{Str}^{\mathsf{O}}_{L}(\mathsf{B})$ from **O-Meet** alone. Taking $\mathcal{Q} \coloneqq \mathsf{B}$ thus yields $\vdash \neg\mathsf{B}\bot \to \forall p(\mathsf{B}p \to p)$, where instantiating $\forall p(\mathsf{B}p \to p)$ at $\bot$ gives the converse. Consistency and factivity of the strongest objective modality are therefore equivalent, and no weaker postulate could deliver the **T** axiom. Assuming instead that some objective modality is trivial $\exists\mathcal{Q}(\mathsf{O}(\mathcal{Q}) \wedge \forall p(\mathcal{Q}p \equiv p))$ is strictly stronger, requiring $Id_{\mathcal{W}} \in \mathcal{C}$ by clause (1) of **L2**, and so failing where $\mathcal{W} \coloneqq \{a, b\}$ and $\mathcal{C} \coloneqq \{\mathcal{W} \times \mathcal{W}\}$ although **T** holds there.[84]

L4 *If* $\mathrm{Str}^{\mathsf{O}}_{L}(\mathcal{Q})$ *and* $\mathrm{Str}^{\mathsf{O}}_{L}(\mathcal{P})$, *then* $\vdash \forall p(\mathcal{Q}p \leftrightarrow \mathcal{P}p)$.

Proof. Instantiating the dominance clause for $\mathcal{Q}$ at $\mathcal{P}$ and detaching with $\vdash \mathsf{O}(\mathcal{P})$ yields $\vdash \forall p(\mathcal{Q}p \to \mathcal{P}p)$. Parity of reasoning yields $\vdash \forall p(\mathcal{P}p \to \mathcal{Q}p)$. □

R3 With existence settled by **T1**, we may ask which operators fall under O, and so coincide with B. One candidate is Cresswell–Suszko's operator $\boxed{\top}\varphi \coloneqq (\varphi \equiv \top)$ which Bacon [6] argues is the broadest necessity, and is supplied here by comprehension **I7** since $p \equiv \top$ contains no operator variables and no occurrence of O.

Moreover, $\boxed{\top}$ is at least as strong as every objective modality. Since instantiating **O-Tran** at $\mathcal{P}$, $\top$, and p gives $\vdash \mathsf{O}(\mathcal{P}) \to ((\top \equiv p) \to (\mathcal{P}\top \equiv \mathcal{P}p))$, it follows that $\vdash \mathsf{O}(\mathcal{P}) \to (\mathcal{P}\top \to (\boxed{\top}p \to \mathcal{P}p))$ by the symmetry of $\equiv$ together with **Imp**. Since **O-Ax** supplies $\mathcal{P}\top$, detaching and generalizing gives $\vdash \forall\mathcal{P}[\mathsf{O}(\mathcal{P}) \to (\boxed{\top} \preceq \mathcal{P})]$. Instantiating

[84] Cf. Williamson [4, p. 457] who assumes some objective modality is trivial, and Bacon and Zeng [67, Prop. 2.2] who provide a derivation.

at B and detaching gives $\boxed{\top} \preceq \mathsf{B}$. The converse would need $\vdash \mathsf{O}(\boxed{\top})$, since **O-Meet** makes B require only what the objective modalities require.

Yet necessitation fails for $\boxed{\top}$, since $\vdash \varphi$ does not yield $\vdash \varphi \equiv \top$, and **K** would require the Boolean identity $(\top \to \psi) \equiv \psi$. Postulating $\mathsf{O}(\boxed{\top})$ would secure both at a price: **O-Nec** then yields $\vdash \varphi \equiv \top$ for every theorem $\vdash \varphi$, the coarsening of $\equiv$ that Bacon's *Rule of Equivalence* delivers and that I have not assumed here.

C1 *If* $\mathrm{Str}^{\mathsf{O}}_L(\Box)$*, then* $\vdash \forall p(\Box p \leftrightarrow \mathsf{B}p)$ *and so* $\Box$ *has an **S5** logic.*

Proof. Existence is ensured by **T1** which shows that B is a strongest objective modal operator. Given $\mathrm{Str}^{\mathsf{O}}_L(\Box)$, **L4** yields $\vdash \forall p(\Box p \leftrightarrow \mathsf{B}p)$, and **T2** yields the **T**, **4**, and **B** axioms for $\Box$ directly. Since $\Box$ inherits **K** by instantiating **O-Ax** and necessitation is given by detaching **O-Nec**, $\Box$ has an **S5** logic as desired. □

5.2 Two-Dimensional Semantics

This section concerns the two-dimensional models introduced in §2.2. Definitions will be restated for convenience throughout.

D9 A *two-dimensional model* is a structure $\mathcal{M} = \langle W, T, \leqslant, |\cdot| \rangle$ where:

Worlds: A nonempty set of *worlds* W.

Times: A nonempty set of *times* T.

Order: A weak total order $\leqslant$ on T.

Interpretation: A function $|\cdot| : \mathbb{L} \to \wp(W \times T)$ assigning each sentence letter to a set of world-time pairs encoding that sentence letter's truth-condition.

D10 The language $\mathcal{L}^{\mathrm{M}} \coloneqq \langle \mathbb{L}, \bot, \to, \boxtimes, \boxed{\scriptstyle P}, \boxed{\scriptstyle F} \rangle$ where $\mathbb{L} \coloneqq \{p_i : i \in \mathbb{N}\}$ is a countable set of sentence letters where the remaining symbols denote falsity, material implication, the disputed modal operator that Thomason reads 'necessarily always', the universal past tense operator, and the universal future tense operator, respectively.

D11 Truth in a two-dimensional model at a world and time is defined recursively:

(p_i) $\mathcal{M}, w, x \vDash p_i$ *iff* $\langle w, x \rangle \in |p_i|$.

$(\bot)$ $\mathcal{M}, w, x \nvDash \bot$.

$(\to)$ $\mathcal{M}, w, x \vDash \varphi \to \psi$ *iff* $\mathcal{M}, w, x \nvDash \varphi$ or $\mathcal{M}, w, x \vDash \psi$.

$(\boxtimes)$ $\mathcal{M}, w, x \vDash \boxtimes\varphi$ *iff* $\mathcal{M}, u, y \vDash \varphi$ for all $u \in W$ and $y \in T$.

$(\boxed{\scriptstyle P})$ $\mathcal{M}, w, x \vDash \boxed{\scriptstyle P}\varphi$ *iff* $\mathcal{M}, w, y \vDash \varphi$ for all $y \in T$ where $y < x$.

$(\boxed{\scriptstyle F})$ $\mathcal{M}, w, x \vDash \boxed{\scriptstyle F}\varphi$ *iff* $\mathcal{M}, w, y \vDash \varphi$ for all $y \in T$ where $x < y$.

D12 A nonempty relation $Z \subseteq (W_1 \times T_1) \times (W_2 \times T_2)$— writing $(w, x) \rightarrow (v, y)$ to represent pairs in Z— is a $\mathcal{L}^{\mathrm{M}}$-*bisimulation* between $\mathcal{M}_1$ and $\mathcal{M}_2$ just in case whenever $(w, x) \rightarrow (w', x')$, all of the following conditions hold:

Atomic harmony: For all $p_i \in \mathbb{L}$, $\mathcal{M}_1, w, x \vDash p_i$ just in case $\mathcal{M}_2, w', x' \vDash p_i$.

Past Forth: For every y with $y < x$ there is y' with $y' < x'$ and $(w, y) \rightarrow (w', y')$.

Past Back: For every y' with $y' < x'$ there is y with $y < x$ and $(w, y) \rightarrow (w', y')$.

Future Forth: For every y with $x < y$ there is y' with $x' < y'$ and $(w, y) \rightarrow (w', y')$.

Future Back: For every y' with $x' < y'$ there is y with $x < y$ and $(w, y) \rightarrow (w', y')$.

Global Forth: For every $(u, z) \in W_1 \times T_1$ there is $(u', z') \in W_2 \times T_2$ with $(u, z) \rightarrow (u', z')$.

Global Back: For every $(u', z') \in W_2 \times T_2$ there is $(u, z) \in W_1 \times T_1$ with $(u, z) \rightarrow (u', z')$.

L5 *If Z is a $\mathcal{L}^{\mathrm{M}}$-bisimulation between $\mathcal{M}_1$ and $\mathcal{M}_2$ and $(w, x) \rightarrow (w', x')$, then for every $\mathcal{L}^{\mathrm{M}}$-formula φ, $\mathcal{M}_1, w, x \vDash \varphi$ just in case $\mathcal{M}_2, w', x' \vDash \varphi$.*

Proof. The proof proceeds by induction on the complexity of φ where the case for the sentence letters and extensional operators are routine.

Case $\boxed{\scriptstyle F}$*:* Assume $\varphi = \boxed{\scriptstyle F}\psi$. Supposing for contraposition that $\mathcal{M}_2, w', x' \nvDash \boxed{\scriptstyle F}\psi$, it follows that $\mathcal{M}_2, w', y' \nvDash \psi$ for some $x' < y'$. By *Future Back*, there is a y with $x < y$ where $(w, y) \rightarrow (w', y')$, and so $\mathcal{M}_1, w, y \nvDash \psi$ by hypothesis. Thus $\mathcal{M}_1, w, x \nvDash \boxed{\scriptstyle F}\psi$. Contraposition and parity of reasoning establish that $\mathcal{M}_1, w, x \vDash \boxed{\scriptstyle F}\psi$ just in case $\mathcal{M}_2, w', x' \vDash \boxed{\scriptstyle F}\psi$. The other tense operators are similar.

Case $\boxtimes$*:* Assume $\varphi = \boxtimes\psi$. Supposing for contraposition that $\mathcal{M}_2, w', x' \nvDash \boxtimes\psi$, it follows that $\mathcal{M}_2, u', z' \nvDash \psi$ for some $u' \in W_2$ and $z' \in T_2$. By *Global Back*, there is $(u, z) \in W_1 \times T_1$ with $(u, z) \rightarrow (u', z')$, and so $\mathcal{M}_1, u, z \nvDash \psi$ by hypothesis. Thus $\mathcal{M}_1, w, x \nvDash \boxtimes\psi$, where contraposition and parity of reasoning complete the case.

By induction on complexity, we may conclude that $\mathcal{M}_1, w, x \vDash \varphi$ just in case $\mathcal{M}_2, w', x' \vDash \varphi$ for all well-formed sentences φ in $\mathcal{L}^{\mathrm{M}}$. □

T3 $\boxed{\scriptstyle 2}$ *is not definable in $\mathcal{L}^{\mathrm{M}}$.*

Proof. Assume for contradiction that $\boxed{\scriptstyle 2}$ is definable in $\mathcal{L}^{\mathrm{M}}$ so that $\boxed{\scriptstyle 2}\varphi$ abbreviates a well-formed sentence of $\mathcal{L}^{\mathrm{M}}$ with the following derived semantic clause:

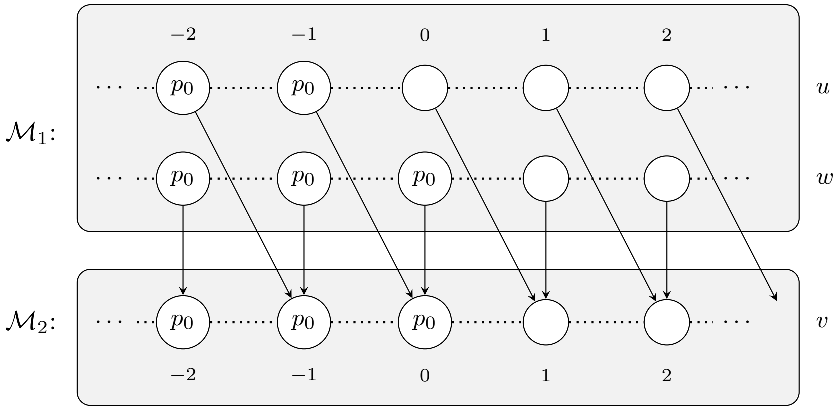
($\boxed{\scriptstyle 2}$) $\mathcal{M}, w, x \vDash \boxed{\scriptstyle 2}\varphi$ *iff* $\mathcal{M}, u, x \vDash \varphi$ for all $u \in W$.

We define the models $\mathcal{M}_1 = \langle W_1, T_1, \leqslant, |\cdot|_1 \rangle$ and $\mathcal{M}_2 = \langle W_2, T_2, \leqslant, |\cdot|_2 \rangle$ where:

$\mathcal{M}_1$: $W_1 = \{w, u\}$, $T_1 = \mathbb{Z}$, and $|p_0|_1 = \{(w, x) : x \leqslant 0\} \cup \{(u, x) : x \leqslant -1\}$, so that w satisfies p_0 at every time up to and including 0, while u satisfies p_0 at every time up to and including -1.

$\mathcal{M}_2$: $W_2 = \{v\}$, $T_2 = \mathbb{Z}$, and $|p_0|_2 = \{(v, x) : x \leqslant 0\}$, so that v satisfies p_0 at every time up to and including 0.

The relation $Z \subseteq (W_1 \times T_1) \times (W_2 \times T_2)$, writing $(w, x) \rightarrow (v, y)$ to represent pairs in Z as before, by $(w, x) \rightarrow (v, x)$ for every $x \in \mathbb{Z}$ and $(u, x) \rightarrow (v, x + 1)$ for every $x \in \mathbb{Z}$. The following diagram depicts a representative window of the construction, where the ellipses indicate that the pattern continues without bound in both directions:



We verify that Z is a $\mathcal{L}^{\mathrm{M}}$-bisimulation by checking the clauses of D12 for an arbitrary pair in Z as given below:

Atomic harmony: For $(w, x) \rightarrow (v, x)$: $\mathcal{M}_1, w, x \vDash p_0$ just in case $x \leqslant 0$, and $\mathcal{M}_2, v, x \vDash p_0$ just in case $x \leqslant 0$. For $(u, x) \rightarrow (v, x + 1)$: $\mathcal{M}_1, u, x \vDash p_0$ just in case $x \leqslant -1$, and $\mathcal{M}_2, v, x + 1 \vDash p_0$ just in case $x + 1 \leqslant 0$, i.e. $x \leqslant -1$. Both pairs agree on p_0 in every case.

Past Forth and Past Back: For $(w, x) \rightarrow (v, x)$, every $z < x$ satisfies $(w, z) \rightarrow (v, z)$ with $z < x$, giving both directions by taking $z' = z$. For $(u, x) \rightarrow (v, x + 1)$, every $z < x$ satisfies $(u, z) \rightarrow (v, z + 1)$ with $z + 1 < x + 1$, giving Past Forth; conversely every $z' < x + 1$ in T_2 satisfies $z' - 1 < x$ with $(u, z' - 1) \rightarrow (v, z')$, giving Past Back.

Future Forth and Future Back: Symmetric to the previous case, reversing each inequality throughout.

Global Forth: Every $(w, x) \in W_1 \times T_1$ appears via $(w, x) \rightarrow (v, x)$, and every (u, x) appears via $(u, x) \rightarrow (v, x + 1)$.

Global Back: Every $(v, x) \in W_2 \times T_2$ is the right-projection of $(w, x) \rightarrow (v, x)$.

Thus Z is a $\mathcal{L}^{\mathrm{M}}$-bisimulation. Given that $(w, 0) \rightarrow (v, 0)$, it follows by **L5** that for all well-formed φ of $\mathcal{L}^{\mathrm{M}}$, the following biconditional holds:

$$\mathcal{M}_1, w, 0 \vDash \varphi \text{ just in case } \mathcal{M}_2, v, 0 \vDash \varphi. \tag{$*$}$$

On the assumption that $\boxed{2}$ is definable in $\mathcal{L}^{\mathrm{M}}$, we may conclude that $\mathcal{M}_1, w, 0 \vDash \boxed{2}p_0$ just in case $\mathcal{M}_2, v, 0 \vDash \boxed{2}p_0$. By the claimed semantic clause for $\boxed{2}$ we have:

$$\begin{aligned} \mathcal{M}_1, w, 0 \vDash \boxed{2}p_0 &\Leftrightarrow \mathcal{M}_1, w, 0 \vDash p_0 \text{ and } \mathcal{M}_1, u, 0 \vDash p_0 \\ \mathcal{M}_2, v, 0 \vDash \boxed{2}p_0 &\Leftrightarrow \mathcal{M}_2, v, 0 \vDash p_0 \end{aligned}$$

Since $\mathcal{M}_1, u, 0 \nvDash p_0$, the first gives $\mathcal{M}_1, w, 0 \nvDash \boxed{2}p_0$, while v being the only world of W_2 gives $\mathcal{M}_2, v, 0 \vDash \boxed{2}p_0$, contradicting $(*)$. Thus $\boxed{2}$ with the derived semantic clause above is not definable in $\mathcal{L}^{\mathrm{M}}$ by *reductio*. □

D13 Dorr and Goodman [30] extend their language over product models by adding a countable set of time variables $\mathcal{V} \coloneqq \{t_i : i \in \mathbb{N}\}$ which may be bound by first-order quantifiers; transposing this extension to $\mathcal{L}^{\mathrm{M}}$ for comparison yields the language $\mathcal{L}^{\mathrm{F}}$. An *assignment* is a function $g : \mathcal{V} \to T$ from time variables in $\mathcal{V}$ to times in T which is used to extend the semantics. Truth in a two-dimensional model at a world and time relative to an assignment extends the semantics for $\mathcal{L}^{\mathrm{M}}$ to include the following:

($\exists t$) $\mathcal{M}, w, x, g \vDash \exists t\, \varphi$ *iff* $\mathcal{M}, w, x, g' \vDash \varphi$ for some g' differing from g at most in t.

(Present) $\mathcal{M}, w, x, g \vDash \mathrm{Present}(t)$ *iff* $g(t) = x$.

L6 *In $\mathcal{L}^{\mathrm{F}}$, $\boxed{2}$ may be defined by $\boxed{2}\varphi \coloneqq \exists t\,[Present(t) \wedge \boxtimes(Present(t) \to \varphi)]$.*

Proof. To establish that $\boxed{2}$ is definable, we show:

$$\begin{aligned} &\mathcal{M}, w, x, g \vDash \exists t\,[\mathrm{Present}(t) \wedge \boxtimes(\mathrm{Present}(t) \to \varphi)] \\ &\quad\Leftrightarrow \text{ there is } g' \text{ with } g'(t) = x \text{ and } \mathcal{M}, u, y, g' \vDash \mathrm{Present}(t) \to \varphi \text{ for all } u \in W, y \in T \\ &\quad\Leftrightarrow \text{ there is } g' \text{ with } g'(t) = x \text{ and } \mathcal{M}, u, x, g' \vDash \varphi \text{ for all } u \in W \\ &\quad\Leftrightarrow \mathcal{M}, u, x, g \vDash \varphi \text{ for all } u \in W \end{aligned}$$

The first equivalence follows from the semantics for $\exists$, $\wedge$, and $\boxtimes$. For the second equivalence, since $g'(t) = x$, we have $\mathcal{M}, u, y, g' \vDash \mathrm{Present}(t) \to \varphi$ just in case either $y \neq x$ or $\mathcal{M}, u, y, g' \vDash \varphi$ with $y = x$. It follows that $\mathcal{M}, u, y, g' \vDash \mathrm{Present}(t) \to \varphi$ for all $u \in W$ and $y \in T$ just in case $\mathcal{M}, u, x, g' \vDash \varphi$ for all $u \in W$. The final equivalence holds because t does not occur free in φ, so g and g' agree on the truth of φ, and so we may choose any t-variant g' of g with $g'(t) = x$ to witness the existential. □

T4 *Given the two-dimensional semantics for $\mathcal{L}^{\mathrm{F}}$, both **P1** and **P2** are invalid.*

Proof. Let $\mathcal{M} = \langle W, T, \leqslant, |\cdot| \rangle$ be a two-dimensional model with worlds $W = \{w, u\}$, times $T = \{0, 1\}$ where $\leqslant$ is the usual order on $\{0, 1\}$, and $|p_0| = \{(w, 0), (u, 0)\}$. Since $(w, 0) \in |p_0|$ and $(u, 0) \in |p_0|$, we have $\mathcal{M}, w, 0, g \vDash p_0$ and $\mathcal{M}, u, 0, g \vDash p_0$ where g is any assignment. It follows by **L6** that $\mathcal{M}, w, 0, g \vDash \boxed{2}p_0$.

However, $\mathcal{M}, w, 0, g \vDash \triangle p_0$ requires $\mathcal{M}, w, y, g \vDash p_0$ for every $y \in T$ with $0 \leqslant y$, which fails at $y = 1$ since $(w, 1) \notin |p_0|$. Thus $\mathcal{M}, w, 0, g \nvDash \boxed{2} p_0 \to \triangle p_0$, invalidating **P1** and with it the equivalent **P2**. $\square$

D14 An *order automorphism* on the structure $\langle T, \leqslant \rangle$ is any bijection $\bar{a} : T \to T$ such that for all $x, y \in T$, we have $x \leqslant y$ just in case $\bar{a}(x) \leqslant \bar{a}(y)$.

D15 Given a two-dimensional model $\mathcal{M} = \langle W, T, \leqslant, |\cdot| \rangle$, the worlds $w, w' \in W$ are *time-shifted from x to y*— written $w \approx^y_x w'$— *iff* there exists an order automorphism $\bar{a} : T \to T$ where $y = \bar{a}(x)$ and for all sentence letters $p_i \in \mathbb{L}$ and times $z \in T$, we have $\langle w, z \rangle \in |p_i|$ just in case $\langle w', \bar{a}(z) \rangle \in |p_i|$.

L7 *If $w \approx^y_x u$ for some $x, y \in T$, then there is some order automorphism $\bar{a} : T \to T$ where $w \approx^{\bar{a}(z)}_z u$ for all $z \in T$.*

Proof. Suppose $w \approx^y_x u$ for $x, y \in T$. By **D15**, $y = \bar{a}(x)$ for some order automorphism $\bar{a} : T \to T$ where the following equivalence holds for all $p_i \in \mathbb{L}$ and $z \in T$:

$$\langle w, z \rangle \in |p_i| \text{ just in case } \langle u, \bar{a}(z) \rangle \in |p_i|. \tag{$*$}$$

Let $z \in T$ be arbitrary. To show $w \approx^{\bar{a}(z)}_z u$, we must exhibit an order automorphism $\bar{b} : T \to T$ satisfying two conditions: (i) $\bar{a}(z) = \bar{b}(z)$, and (ii) for all $p_i \in \mathbb{L}$ and $t \in T$, $\langle w, t \rangle \in |p_i|$ just in case $\langle u, \bar{b}(t) \rangle \in |p_i|$. Taking $\bar{b} = \bar{a}$ satisfies both conditions: (i) holds trivially, and (ii) is exactly $(*)$. Since z was arbitrary, $w \approx^{\bar{a}(z)}_z u$ for all $z \in T$. $\square$

D16 A two-dimensional model $\mathcal{M} = \langle W, T, \leqslant, |\cdot| \rangle$ of $\mathcal{L}^{\text{K}}$ is *abundant* just in case for every $w \in W$ and times $x, y \in T$, there is some world $w' \in W$ where $w \approx^y_x w'$.

L8 *$\mathcal{M}_2, w, x \vDash \varphi$ just in case $\mathcal{M}_2, u, y \vDash \varphi$ for any well-formed sentence φ of $\mathcal{L}^{\text{K}}$ and abundant two-dimensional model $\mathcal{M}_2 = \langle W, T, \leqslant, |\cdot| \rangle$ where $w \approx^y_x u$.*

Proof. Assume $w \approx^y_x u$ in $\mathcal{M}_2 = \langle W, T, \leqslant, |\cdot| \rangle$. The proof proceeds by induction on the complexity of φ. By **D15**, there exists an order automorphism $\bar{a} : T \to T$ where $y = \bar{a}(x)$ and for all sentence letters $p_i \in \mathbb{L}$ and times $z \in T$:

$$\langle w, z \rangle \in |p_i| \;\Leftrightarrow\; \langle u, \bar{a}(z) \rangle \in |p_i|. \tag{$\dagger$}$$

Base Case ($\varphi = p_i$): By $(\dagger)$ with $z = x$, it follows that $\langle w, x \rangle \in |p_i|$ just in case $\langle u, \bar{a}(x) \rangle \in |p_i|$. Since $y = \bar{a}(x)$, this gives $\langle w, x \rangle \in |p_i|$ just in case $\langle u, y \rangle \in |p_i|$. By the semantic clause for sentence letters, $\mathcal{M}_2, w, x \vDash p_i$ just in case $\mathcal{M}_2, u, y \vDash p_i$.

Base Case ($\varphi = \bot$): By the semantic clause for $\bot$, we have $\mathcal{M}_2, w, x \nvDash \bot$ and $\mathcal{M}_2, u, y \nvDash \bot$. Therefore the biconditional holds immediately.

Inductive Case ($\varphi = \psi \to \chi$): immediate from the inductive hypothesis.

Inductive Case ($\varphi = \Box\psi$):

$$\begin{aligned}
\mathcal{M}_2, w, x \not\models \Box\psi \;&\Leftrightarrow\; \mathcal{M}_2, v, x \not\models \psi \text{ for some } v \in W \\
&\Leftrightarrow\; \mathcal{M}_2, v', y \not\models \psi \text{ for some } v \in W \\
&\Leftrightarrow\; \mathcal{M}_2, u, y \not\models \Box\psi
\end{aligned}$$

Since $\mathcal{M}_2$ is abundant where $v \in W$ and $x, y \in T$, there is some $v' \in W$ where $v \approx^y_x v'$. It follows by the induction hypothesis that $\mathcal{M}_2, v', y \not\models \psi$, where parity of reasoning establishes the converse. This justifies the second biconditional.

Inductive Case ($\varphi = \boxed{\mathrm{P}}\psi$):

$$\begin{aligned}
\mathcal{M}_2, w, x \not\models \boxed{\mathrm{P}}\psi \;&\Leftrightarrow\; \mathcal{M}_2, w, z \not\models \psi \text{ for some } z < x \\
&\Leftrightarrow\; \mathcal{M}_2, u, z' \not\models \psi \text{ for some } z' < y \\
&\Leftrightarrow\; \mathcal{M}_2, u, y \not\models \boxed{\mathrm{P}}\psi
\end{aligned}$$

For the forward direction of the second biconditional, assume that $\mathcal{M}_2, w, z \not\models \psi$ for some $z < x$. Since $\bar{a}$ is an order automorphism, $\bar{a}(z) < \bar{a}(x)$ where $\bar{a}(x) = y$. Letting $z' = \bar{a}(z)$, we have $z' < y$. Since $w \approx^y_x u$ where $y = \bar{a}(x)$ and $z' = \bar{a}(z)$, it follows by **L7** that $w \approx^{z'}_z u$. Applying the induction hypothesis to ψ, we have $\mathcal{M}_2, w, z \models \psi$ just in case $\mathcal{M}_2, u, z' \models \psi$, and so $\mathcal{M}_2, u, z' \not\models \psi$ for some $z' < y$. The backward direction runs the same argument through $\bar{a}^{-1}$, which is an order automorphism carrying y to x, so that any $z' < y$ yields $z \coloneqq \bar{a}^{-1}(z') < x$ with $w \approx^{z'}_z u$ as before.

Inductive Case ($\varphi = \boxed{\mathrm{F}}\psi$): The proof is analogous to the case for $\boxed{\mathrm{P}}\psi$, using $x < z$ and $y < z'$ in place of $z < x$ and $z' < y$. $\Box$

T5 *Both **P1** and **P2** are valid over a two-dimensional frame $\langle W, T, \leqslant\rangle$ with $W \neq \varnothing$ if and only if $|T| \leqslant 1$.*

Proof. ($\Leftarrow$) Suppose $|T| \leqslant 1$. Since every two-dimensional model requires T to be nonempty (**D9**), we have $|T| = 1$; let $T = \{x\}$. Then $\boxed{\mathrm{F}}\varphi$ requires $\mathcal{M}, w, y \models \varphi$ for all y with $x < y$, and $\boxed{\mathrm{P}}\varphi$ requires $\mathcal{M}, w, y \models \varphi$ for all y with $y < x$; since $T = \{x\}$ contains no such y in either case, both $\boxed{\mathrm{F}}\varphi$ and $\boxed{\mathrm{P}}\varphi$ are vacuously true at $\mathcal{M}, w, x$ for every φ. Since $\triangle\varphi \coloneqq \boxed{\mathrm{P}}\varphi \wedge \varphi \wedge \boxed{\mathrm{F}}\varphi$, it follows that $\mathcal{M}, w, x \models \triangle\varphi$ just in case $\mathcal{M}, w, x \models \varphi$. Given that $\Box\varphi$ entails $\mathcal{M}, u, x \models \varphi$ for all $u \in W$, we have in particular $\mathcal{M}, w, x \models \varphi$, and so $\mathcal{M}, w, x \models \triangle\varphi$. Thus $\Box\varphi \rightarrow \triangle\varphi$ is valid, and **P2** is equivalent.

($\Rightarrow$) Suppose $|T| \geqslant 2$, so that there are distinct $x, y \in T$ with $x < y$ since $\leqslant$ is a weak total order. Let $w \in W$ be arbitrary and define $|p_0| = W \times \{x\}$, so that $\mathcal{M}, u, x \models p_0$ for every $u \in W$ and hence $\mathcal{M}, w, x \models \Box p_0$ by the semantic clause for $\Box$. But $(w, y) \notin |p_0|$ where $x < y$, and so $\mathcal{M}, w, x \not\models \boxed{\mathrm{F}}p_0$ and hence $\mathcal{M}, w, x \not\models \triangle p_0$. Thus $\mathcal{M}, w, x \not\models \Box p_0 \rightarrow \triangle p_0$, falsifying **P1** and with it the equivalent **P2**. $\Box$

T6 *Both **P1** and **P2** are valid over the abundant two-dimensional models of $\mathcal{L}^{\mathrm{K}}$.*

Proof. Let $\mathcal{M} = \langle W, T, \leqslant, |\cdot| \rangle$ be an abundant two-dimensional model of $\mathcal{L}^{\mathrm{K}}$. Assume for *reductio* that $\mathcal{M}, w, x \nvDash \Box\varphi \to \triangle\varphi$ for some $w \in W$, $x \in T$, and well-formed sentence φ of $\mathcal{L}^{\mathrm{K}}$. Thus $\mathcal{M}, w, x \vDash \Box\varphi$ and $\mathcal{M}, w, x \nvDash \triangle\varphi$.

By the semantic clause for $\Box$, we have $\mathcal{M}, u, x \vDash \varphi$ for all $u \in W$. In particular, $\mathcal{M}, w, x \vDash \varphi$. Given that $\triangle\varphi \coloneqq \boxed{\mathrm{P}}\varphi \wedge \varphi \wedge \boxed{\mathrm{F}}\varphi$ and $\mathcal{M}, w, x \vDash \varphi$, it follows from $\mathcal{M}, w, x \nvDash \triangle\varphi$ that either $\mathcal{M}, w, x \nvDash \boxed{\mathrm{P}}\varphi$ or $\mathcal{M}, w, x \nvDash \boxed{\mathrm{F}}\varphi$.

Case 1: Assume $\mathcal{M}, w, x \nvDash \boxed{\mathrm{P}}\varphi$. By the semantic clause for $\boxed{\mathrm{P}}$, there exists $y \in T$ with $y < x$ such that $\mathcal{M}, w, y \nvDash \varphi$. Since $\mathcal{M}$ is abundant, there exists $w' \in W$ such that $w \approx^x_y w'$. By **L8**, $\mathcal{M}, w, y \vDash \varphi$ just in case $\mathcal{M}, w', x \vDash \varphi$. Since $\mathcal{M}, w, y \nvDash \varphi$, we have $\mathcal{M}, w', x \nvDash \varphi$. This contradicts that $\mathcal{M}, u, x \vDash \varphi$ for all $u \in W$.

Case 2: Assume $\mathcal{M}, w, x \nvDash \boxed{\mathrm{F}}\varphi$. By the semantic clause for $\boxed{\mathrm{F}}$, there exists $z \in T$ with $x < z$ such that $\mathcal{M}, w, z \nvDash \varphi$. Since $\mathcal{M}$ is abundant, there exists $w'' \in W$ such that $w \approx^x_z w''$. By **L8**, $\mathcal{M}, w, z \vDash \varphi$ just in case $\mathcal{M}, w'', x \vDash \varphi$. Since $\mathcal{M}, w, z \nvDash \varphi$, we have $\mathcal{M}, w'', x \nvDash \varphi$. This contradicts that $\mathcal{M}, u, x \vDash \varphi$ for all $u \in W$.

Both cases yield a contradiction. Therefore $\mathcal{M}, w, x \vDash \Box\varphi \to \triangle\varphi$ for all abundant models $\mathcal{M}$, worlds $w \in W$, times $x \in T$, and well-formed sentences φ of $\mathcal{L}^{\mathrm{K}}$. Hence **P1** is valid over the abundant two-dimensional models of $\mathcal{L}^{\mathrm{K}}$.

Since $\Box\neg\varphi \to \triangle\neg\varphi$ by **P1**, contraposition yields $\neg\triangle\neg\varphi \to \neg\Box\neg\varphi$. By definition, $\triangledown\varphi \to \Diamond\varphi$, and so **P2** is valid over the abundant two-dimensional models of $\mathcal{L}^{\mathrm{K}}$. $\square$

T7 *Abundant models with at least two distinct times are unbounded.*

Proof. Let $\mathcal{M} = \langle W, T, \leqslant, |\cdot| \rangle$ be an abundant two-dimensional model of $\mathcal{L}^{\mathrm{K}}$ where $x < y$ for $x, y \in T$. We prove that T is unbounded below and unbounded above.

Below: Assume for *reductio* that T is bounded below. Thus there is some $t_{\min} \in T$ where $t_{\min} \leqslant z$ for all $z \in T$. Since $t_{\min} \leqslant x < y$, either $t_{\min} < x$ or $t_{\min} = x < y$. In either case, there exists $t \in \{x, y\}$ with $t_{\min} < t$. Letting $w \in W$, it follows by abundance that there is some $w' \in W$ such that $w \approx^t_{t_{\min}} w'$. By definition, there is an order automorphism $\bar{a} : T \to T$ where $t = \bar{a}(t_{\min})$. Since $t_{\min} \leqslant z$ for all $z \in T$ and $\bar{a}$ is order-preserving, we have $\bar{a}(t_{\min}) \leqslant \bar{a}(z)$ for all $z \in T$. Since $\bar{a}$ is surjective, for any $z' \in T$ there exists $z \in T$ where $\bar{a}(z) = z'$. Thus $\bar{a}(t_{\min}) \leqslant z'$ for all $z' \in T$, so $\bar{a}(t_{\min})$ is minimal in T. Therefore $\bar{a}(t_{\min}) = t_{\min}$, and so $t = t_{\min}$, which contradicts $t_{\min} < t$. We may conclude by *reductio* that T is unbounded below.

Above: The argument is the mirror image, reversing every inequality: a maximum $t_{\max}$ yields some $t \in \{x, y\}$ with $t < t_{\max}$, abundance and **D15** supply an order automorphism carrying $t_{\max}$ to t, and an order automorphism of T fixes a maximum as surely as it fixes a minimum, contradicting $t < t_{\max}$. $\square$

5.3 Task Semantics

The definitions of **§3** are restated here for the language $\mathcal{L}$ deriving the basic results in the form in which they are formalized in the Lean 4 repository for this paper.

D17 A *temporal order* is a nontrivial totally ordered abelian group $\mathcal{D} = \langle D, +, 0, \leqslant \rangle$ with *positive cone* $D^+ \coloneqq \{x \in D : x \geqslant 0\}$.

D18 A *task relation* on a nonempty set of *world states* W over a temporal order $\mathcal{D}$ is any parameterized relation $w \Rightarrow_x u$ for $w, u \in W$ and $x \in D^+$, extended to negative durations by the *reflection convention* $w \Rightarrow_{-x} u \coloneqq u \Rightarrow_x w$ for $x \geqslant 0$, determining the following for any world states $w, v \in W$ and durations $x, y \in D$:

Fiber: $\texttt{fib}(w, x) \coloneqq \{u \in W : w \Rightarrow_x u\}$.

Cone: $(w)_x \coloneqq \bigcup_{|y|<x} \texttt{fib}(w, y)$ where $x > 0$.

Segment: $[w, v]_x^y \coloneqq \texttt{fib}(w, x) \cap \texttt{fib}(v, -y)$ where $x, y \geqslant 0$.

D19 Letting a nonempty family of sets $\mathcal{S}$ be $\supseteq$-*directed* just in case $S \subseteq S_1 \cap S_2$ for some $S \in \mathcal{S}$ whenever $S_1, S_2 \in \mathcal{S}$, a *task frame* is any $\mathcal{F} = \langle W, \mathcal{D}, \Rightarrow \rangle$ where W is a nonempty set of world states, $\mathcal{D}$ is a temporal order, and $\Rightarrow$ is a task relation satisfying the following for all positive durations $x, y \geqslant 0$:

Compositionality: $w \Rightarrow_{x+y} v$ if and only if $w \Rightarrow_x u$ and $u \Rightarrow_y v$ for some $u \in W$.

Seriality: $w \Rightarrow_x u$ and $v \Rightarrow_x w$ for some $u, v \in W$.

Limit: $\bigcap_{x>0} (w)_x = \{w\}$.

Saturation: $\bigcap \mathcal{S} \neq \varnothing$ for any $\supseteq$-directed family $\mathcal{S}$ of nonempty fibers and segments.[85]

L9 *$w \Rightarrow_0 w$ for every world state $w \in W$ in every task frame $\mathcal{F} = \langle W, \mathcal{D}, \Rightarrow \rangle$.*

Proof. By *Seriality* at $x = 0$, there is some $u \in W$ where $w \Rightarrow_0 u$. Since $|0| < x$ for every $x > 0$, it follows that $u \in (w)_x$ for every $x > 0$. Thus $u \in \bigcap_{x>0}(w)_x = \{w\}$ by *Limit*, and so $u = w$, whence $w \Rightarrow_0 w$. □

L10 *Every task frame $\mathcal{F} = \langle W, \mathcal{D}, \Rightarrow \rangle$ with finite W satisfies Saturation, choice-free.*

Proof. Let $\mathcal{S}$ be a $\supseteq$-directed family of nonempty fibers and segments, as in **D19**. Since W is finite, every member of $\mathcal{S}$ is a subset of W, so $\mathcal{S}$ has finitely many distinct members. Downward $\supseteq$-directedness gives, for any two members of $\mathcal{S}$, some member of $\mathcal{S}$ contained in their intersection, and iterating this over the finitely many members yields a least member $S^* \in \mathcal{S}$ with $S^* \subseteq S$ for every $S \in \mathcal{S}$. Since S^* is itself a member of $\mathcal{S}$, it is nonempty, and $S^* \subseteq \bigcap \mathcal{S} \subseteq S^*$, so $\bigcap \mathcal{S} = S^* \neq \varnothing$. □

D20 Given a task frame $\mathcal{F} = \langle W, \mathcal{D}, \Rightarrow \rangle$, define:

Basic Opens: $B_{\mathcal{F}} \coloneqq \{(w)_x : w \in W \text{ and } x \in D \text{ with } x > 0\}$.

[85] The nonempty fibers and segments form a *ball space* on W in the sense of Ćmiel, Kuhlmann, and Kuhlmann [68]. *Saturation* is the downward-directed-intersection condition $\mathbf{S}_1^d$ of the ball-space hierarchy— the nest condition $\mathbf{S}_1$ with a $\supseteq$-directed system of balls in place of a nest— and so is at least as strong as the standard *spherically complete* condition, which is $\mathbf{S}_1$ itself.

Topology: $\mathcal{T}_{\mathcal{F}} := \langle W, \mathcal{O}_{\mathcal{F}} \rangle$ where $\mathcal{O}_{\mathcal{F}}$ is the result of closing $B_{\mathcal{F}}$ under arbitrary union and finite intersection.

Closure: $\overline{S} := \{w \in W : O \cap S \neq \varnothing$ for every open $O \in \mathcal{O}_{\mathcal{F}}$ where $w \in O\}$ for $S \subseteq W$.

T1: A topology is *T1* just in case $\overline{\{w\}} = \{w\}$ for all $w \in W$.

R0: A topology is *R0* just in case $w \in \overline{\{u\}}$ iff $u \in \overline{\{w\}}$ for all $w, u \in W$.

T8 *$\mathcal{T}_{\mathcal{F}}$ is T1 for every task frame $\mathcal{F}$.*

Proof. Let $\mathcal{F} = \langle W, \mathcal{D}, \Rightarrow \rangle$ be a task frame and $u \in W$. Since $u \in O \cap \{u\}$ for every open $O \in \mathcal{O}_{\mathcal{F}}$ containing u, it follows by *Closure* in **D20** that $\{u\} \subseteq \overline{\{u\}}$.

For the converse inclusion, suppose $w \in \overline{\{u\}}$, so that $u \in O$ for every open $O \in \mathcal{O}_{\mathcal{F}}$ containing w by *Closure*. Since $w \Rightarrow_0 w$ by **L9**, each basic open $(w)_x$ with $x > 0$ contains w, and so $u \in (w)_x$ by the above. Thus for every $x > 0$, there is some $y \in D$ where $|y| < x$ and $w \Rightarrow_y u$, and so $u \Rightarrow_{-y} w$ by the reflection convention, whence $w \in (u)_x$ for every $x > 0$. Thus $w \in \bigcap_{x>0}(u)_x = \{u\}$ by *Limit*, and so $w = u$, establishing $\overline{\{u\}} \subseteq \{u\}$. Hence $\overline{\{u\}} = \{u\}$ for every $u \in W$, and so $\mathcal{T}_{\mathcal{F}}$ is *T1* by **D20**. $\square$

C2 *$\mathcal{T}_{\mathcal{F}}$ is R0 for every task frame $\mathcal{F}$.*

Proof. Given $w \in \overline{\{u\}}$, **T8** yields $\overline{\{u\}} = \{u\}$, so $w = u$, whence $u \in \overline{\{w\}}$ by **T8** again. The converse direction is symmetric, and so $\mathcal{T}_{\mathcal{F}}$ is *R0* by **D20**. $\square$

D21 A *partial history* over a task frame $\mathcal{F} = \langle W, \mathcal{D}, \Rightarrow \rangle$ is a function $\tau : X \to W$ on a nonempty set $X \subseteq D$ where $\tau(x) \Rightarrow_{y-x} \tau(y)$ for all times $x, y \in X$. A *convex history* is any partial history whose domain X is *convex*, so that $y \in X$ whenever $x, z \in X$ and $x < y < z$. A *possible world* is any convex history whose domain is total, so that $X = D$. A partial history σ *extends* τ just in case $\mathtt{dom}(\tau) \subseteq \mathtt{dom}(\sigma)$ and $\tau(x) = \sigma(x)$ for all $x \in \mathtt{dom}(\tau)$. The set of all possible worlds over $\mathcal{F}$ is denoted $H_{\mathcal{F}}$.

D22 For a partial history $\tau : X \to W$ over a task frame $\mathcal{F}$ and duration $z \in D \backslash X$, the *constraints on* z are the segments $[\tau(t), \tau(s)]^{s-z}_{z-t}$ for times $t, s \in X$ where $t < z < s$ when both $t, s \in X$, and the fibers $\mathtt{fib}(\tau(t), z - t)$ for $t \in X$ otherwise.

L11 *For any partial history $\tau : X \to W$ over a task frame $\mathcal{F} = \langle W, \mathcal{D}, \Rightarrow \rangle$ and duration $z \in D \backslash X$, the fibers $\mathtt{fib}(\tau(t'), z - t') \subseteq \mathtt{fib}(\tau(t), z - t)$ nest for all times $t \leqslant t' < z$ in X and symmetrically for all times $z < t' \leqslant t$ in X, while the segments $[\tau(t'), \tau(s')]^{s'-z}_{z-t'} \subseteq [\tau(t), \tau(s)]^{s-z}_{z-t}$ nest for all times $t \leqslant t' < z < s' \leqslant s$ in X.*

Proof. Suppose $t \leqslant t' < z$ in X and consider any $u \in \mathtt{fib}(\tau(t'), z - t')$, so that $\tau(t') \Rightarrow_{z-t'} u$. Since τ is a partial history, $\tau(t) \Rightarrow_{t'-t} \tau(t')$, and so $\tau(t) \Rightarrow_{z-t} u$ by *Compositionality*, where $z - t = (t' - t) + (z - t')$, whence $u \in \mathtt{fib}(\tau(t), z - t)$.

Since the case where $z < t' \leqslant t$ in X is similar, assume $t \leqslant t' < z < s' \leqslant s$ in X and consider any $u \in [\tau(t'), \tau(s')]^{s'-z}_{z-t'}$, so that $u \in \mathtt{fib}(\tau(t'), z - t')$ and $u \in \mathtt{fib}(\tau(s'), z - s')$.

Since $t \leqslant t' < z$, the first inclusion above gives $u \in \mathtt{fib}(\tau(t), z-t)$, and since $z < s' \leqslant s$, its symmetric case gives $u \in \mathtt{fib}(\tau(s), z-s)$. Thus $u \in [\tau(t), \tau(s)]^{s-z}_{z-t}$ as desired. □

L12 *For any partial history $\tau : X \to W$ over a task frame $\mathcal{F} = \langle W, \mathcal{D}, \Rightarrow \rangle$ and duration $z \in D \backslash X$, every constraint imposed on z is nonempty.*

Proof. A fiber $\mathtt{fib}(\tau(t), z-t)$ imposed on z is nonempty by *Seriality*: when $t < z$, some $u \in W$ satisfies $\tau(t) \Rightarrow_{z-t} u$, and when $t > z$, some $u \in W$ satisfies $u \Rightarrow_{t-z} \tau(t)$, which is $\tau(t) \Rightarrow_{z-t} u$ by the reflection convention. A segment $[\tau(t), \tau(s)]^{s-z}_{z-t}$ imposed on z for times $t < z < s$ is nonempty by *Compositionality*: since τ is a partial history, $\tau(t) \Rightarrow_{s-t} \tau(s)$ where $s - t = (z-t) + (s-z)$, and so *Compositionality* provides some $u \in W$ where $\tau(t) \Rightarrow_{z-t} u$ and $u \Rightarrow_{s-z} \tau(s)$, whence $u \in [\tau(t), \tau(s)]^{s-z}_{z-t}$. □

L13 *For any partial history $\tau : X \to W$ over a task frame $\mathcal{F}$ and duration $z \in D \backslash X$, the constraints imposed on z form a $\supseteq$-directed family of nonempty sets.*

Proof. The family is nonempty since X is nonempty, and every member is nonempty by **L12**, so it remains to establish $\supseteq$-directedness. Split X into $A \coloneqq \{t \in X : t < z\}$ and $C \coloneqq \{s \in X : s > z\}$, so that by **D22** the constraints imposed on z are the segments $S_{t,s} \coloneqq [\tau(t), \tau(s)]^{s-z}_{z-t}$ for pairs $\langle t, s \rangle \in A \times C$ when A and C are both nonempty, and the fibers $F_t \coloneqq \mathtt{fib}(\tau(t), z-t)$ for $t \in X$ otherwise.

For segments $S_{t,s}$ and $S_{t',s'}$, the times $t'' \coloneqq \max\{t, t'\}$ and $s'' \coloneqq \min\{s, s'\}$ belong to A and C respectively, so $S_{t'',s''} \subseteq S_{t,s} \cap S_{t',s'}$ is a member of the family by **L11**. For fibers F_t and $F_{t'}$, exactly one of A and C is empty by **D22**, since $X = A \cup C$ is nonempty, so t and t' lie on the same side of z. Assume that $X = A$ and take $t'' \coloneqq \max\{t, t'\}$, so that $t \leqslant t'' < z$ and $t' \leqslant t'' < z$ in X. Thus $F_{t''} \subseteq F_t \cap F_{t'}$ is a member of the family by **L11**. The case where $X = C$ is symmetric. □

L14 *For any partial history $\tau : X \to W$ over a task frame $\mathcal{F} = \langle W, \mathcal{D}, \Rightarrow \rangle$ and duration $z \in D \backslash X$, a world state belongs to every member of the constraints imposed on z just in case it belongs to every fiber $\mathtt{fib}(\tau(t), z-t)$ for $t \in X$.*

Proof. Write $F_t \coloneqq \mathtt{fib}(\tau(t), z-t)$ for $t \in X$ and let $\mathcal{C}$ be the family of constraints imposed on z. Split X into $A \coloneqq \{t \in X : t < z\}$ and $C \coloneqq \{s \in X : s > z\}$, which exhaust X since $z \notin X$, where X is nonempty by **D21**. Should one of A and C be empty, **D22** makes $\mathcal{C} = \{F_t : t \in X\}$ outright, whence $\bigcap \mathcal{C} = \bigcap_{t \in X} F_t$ immediately. Otherwise A and C are both nonempty, where $[\tau(t), \tau(s)]^{s-z}_{z-t} = F_t \cap F_s$ by **D18**, and so **D22** makes $\mathcal{C} = \{F_t \cap F_s : \langle t, s \rangle \in A \times C\}$. As each member of A and of C occurs in some such pair, $\bigcap \mathcal{C} = \bigcap_{\langle t,s \rangle \in A \times C} (F_t \cap F_s) = \bigcap_{t \in A \cup C} F_t = \bigcap_{t \in X} F_t$ as desired. □

L15 *For any partial history $\tau : X \to W$ over a task frame $\mathcal{F} = \langle W, \mathcal{D}, \Rightarrow \rangle$ and duration $z \in D \backslash X$, the function $\tau \cup \{\langle z, u \rangle\}$ is a partial history on $X \cup \{z\}$ just in case u belongs to every member of the constraints imposed on z.*

Proof. Write $\sigma \coloneqq \tau \cup \{\langle z, u \rangle\}$ and $F_t \coloneqq \mathtt{fib}(\tau(t), z-t)$ for $t \in X$. By **D21**, σ is a partial history on $X \cup \{z\}$ just in case $\sigma(x) \Rightarrow_{y-x} \sigma(y)$ for every pair of times $x, y \in X \cup \{z\}$, and these pairs are of three kinds. The pairs drawn wholly from X demand no more than what already holds of τ. The pair of z with itself demands the zero loop $u \Rightarrow_0 u$, which holds by **L9**. Each remaining pair consists of z and some $t \in X$, demanding $\tau(t) \Rightarrow_{z-t} u$ in either order under the reflection convention, which is to say $u \in F_t$. So σ is a partial history on $X \cup \{z\}$ just in case u belongs to every fiber F_t for $t \in X$, which by **L14** is to belong to every member of the constraints imposed on z. □

L16 *Every partial history $\tau : X \to W$ over a task frame $\mathcal{F} = \langle W, \mathcal{D}, \Rightarrow\rangle$ extends to a partial history on $X \cup \{z\}$ for any duration $z \in D$.*

Proof. If $z \in X$, then τ extends itself. Otherwise the constraints imposed on z form a $\supseteq$-directed family of nonempty fibers and segments by **L13**, and so *Saturation* provides some $u \in W$ belonging to every member, whence $\tau \cup \{\langle z, u\rangle\}$ is a partial history on $X \cup \{z\}$ extending τ by **L15**. When the family has a $\subseteq$-least member— as **L11** provides whenever X contains an assignment nearest to z on each side it occupies— that member already contains a candidate and *Saturation* is not needed. □

T9 *Every partial history $\tau : X \to W$ over a task frame $\mathcal{F} = \langle W, \mathcal{D}, \Rightarrow\rangle$ is extended by some possible world $\sigma \in H_{\mathcal{F}}$.*[86]

Proof. The partial histories extending τ are partially ordered by extension, and every chain among them is bounded above by its union, which restricts on any pair of times to a single member of the chain and so is itself a partial history. By Zorn's lemma, there is a maximal partial history $\sigma : T \to W$ extending τ. If $T \neq D$, then σ extends to a partial history on $T \cup \{z\}$ for any $z \in D \backslash T$ by **L16**, contradicting maximality. Thus $T = D$, whence $\sigma \in H_{\mathcal{F}}$ is a possible world extending τ by **D21**. □

C3 *For any task frame $\mathcal{F} = \langle W, \mathcal{D}, \Rightarrow\rangle$, world state $w \in W$, and time $x \in D$, there is a possible world $\tau \in H_{\mathcal{F}}$ where $\tau(x) = w$, and so $H_{\mathcal{F}} \neq \varnothing$.*

Proof. Since $w \Rightarrow_0 w$ by **L9**, the function $\{\langle x, w\rangle\}$ is a partial history on the nonempty set $\{x\}$. By **T9**, some possible world $\tau \in H_{\mathcal{F}}$ extends it, where $\tau(x) = w$. Since W is nonempty, $H_{\mathcal{F}} \neq \varnothing$. □

D23 The language $\mathcal{L} \coloneqq \langle \mathbb{L}, \bot, \to, \Box, \lhd, \rhd\rangle$ where $\mathbb{L} \coloneqq \{p_i : i \in \mathbb{N}\}$ is a countable set of *sentence letters* with operators for *falsity*, *material implication*, *metaphysical necessity*, *since*, and *until*, respectively. Well-formed sentences of $\mathcal{L}$ are defined by:

$$\varphi, \psi ::= p_i \mid \bot \mid \varphi \to \psi \mid \Box\varphi \mid \varphi \lhd \psi \mid \varphi \rhd \psi.$$

The following operators are defined in $\mathcal{L}$:

[86] The proof appeals to Zorn's lemma, and so the derivation of *Occurrence* from *Seriality* and *Saturation* in **C3** is a theorem of ZFC, in contrast with the derivation of the zero loops in **L9** and the derivation of *Saturation* for finite W in **L10**, both of which are choice-free.

Some Past: $\diamondsuit_{P}\varphi \coloneqq \top \lhd \varphi$.

Some Future: $\diamondsuit_{F}\varphi \coloneqq \top \rhd \varphi$.

All Past: $\boxed{\scriptstyle P}\varphi \coloneqq \neg\diamondsuit_{P}\neg\varphi$.

All Future: $\boxed{\scriptstyle F}\varphi \coloneqq \neg\diamondsuit_{F}\neg\varphi$.

Always: $\triangle\varphi \coloneqq \boxed{\scriptstyle P}\varphi \wedge \varphi \wedge \boxed{\scriptstyle F}\varphi$.

Sometimes: $\triangledown\varphi \coloneqq \diamondsuit_{P}\varphi \vee \varphi \vee \diamondsuit_{F}\varphi$.

Next: $\textcircled{\scriptsize F}\varphi \coloneqq \bot \rhd \varphi$.

Previous: $\textcircled{\scriptsize P}\varphi \coloneqq \bot \lhd \varphi$.

D24 A *model* of $\mathcal{L}$ is a structure $\mathcal{M} = \langle W, \mathcal{D}, \Rightarrow, |\cdot|\rangle$ where $\mathcal{F} = \langle W, \mathcal{D}, \Rightarrow\rangle$ is a task frame and $|p_i| \subseteq W$ for every sentence letter $p_i \in \mathbb{L}$. Relative to a model $\mathcal{M}$, possible world $\tau \in H_{\mathcal{F}}$, and time $x \in D$, *truth* is defined recursively as follows:

(p_i) $\mathcal{M}, \tau, x \vDash p_i$ *iff* $\tau(x) \in |p_i|$.

($\bot$) $\mathcal{M}, \tau, x \nvDash \bot$.

($\to$) $\mathcal{M}, \tau, x \vDash \varphi \to \psi$ *iff* $\mathcal{M}, \tau, x \nvDash \varphi$ or $\mathcal{M}, \tau, x \vDash \psi$.

($\Box$) $\mathcal{M}, \tau, x \vDash \Box\varphi$ *iff* $\mathcal{M}, \sigma, x \vDash \varphi$ for all $\sigma \in H_{\mathcal{F}}$.

($\lhd$) $\mathcal{M}, \tau, x \vDash \varphi \lhd \psi$ *iff* $\mathcal{M}, \tau, z \vDash \psi$ for some time $z < x$ where $\mathcal{M}, \tau, y \vDash \varphi$ for all $y \in D$ with $z < y < x$.

($\rhd$) $\mathcal{M}, \tau, x \vDash \varphi \rhd \psi$ *iff* $\mathcal{M}, \tau, z \vDash \psi$ for some time $z > x$ where $\mathcal{M}, \tau, y \vDash \varphi$ for all $y \in D$ with $x < y < z$.

Since the defined operators $\boxed{\scriptstyle P}$ and $\boxed{\scriptstyle F}$ satisfy the natural truth clauses, proofs will appeal to them freely without derivation.

D25 For a task frame $\mathcal{F} = \langle W, \mathcal{D}, \Rightarrow\rangle$, the possible worlds $\tau, \sigma \in H_{\mathcal{F}}$ are *time-shifted from x to y*— written $\tau \approx^y_x \sigma$— *iff* $\tau(z) = \sigma(z + y - x)$ for all $z \in D$.

L17 *For any task frame $\mathcal{F} = \langle W, \mathcal{D}, \Rightarrow\rangle$, possible world $\tau \in H_{\mathcal{F}}$, and times $x, y \in D$, there is a possible world $\sigma \in H_{\mathcal{F}}$ where $\tau \approx^y_x \sigma$.*

Proof. Let $\mathcal{F} = \langle W, \mathcal{D}, \Rightarrow\rangle$ be a task frame, $\tau \in H_{\mathcal{F}}$ a possible world where $x, y \in D$ are arbitrary times. Define $\sigma(z) \coloneqq \tau(z + x - y)$ for all $z \in D$, which is total since $\mathcal{D}$ is a group. Given any $z_1, z_2 \in D$, we have $(z_2 + x - y) - (z_1 + x - y) = z_2 - z_1$. Since τ is a possible world, $\tau(z_1 + x - y) \Rightarrow_{z_2 - z_1} \tau(z_2 + x - y)$, and so $\sigma(z_1) \Rightarrow_{z_2 - z_1} \sigma(z_2)$, whence $\sigma \in H_{\mathcal{F}}$. Finally, $\sigma(z + y - x) = \tau(z + y - x + x - y) = \tau(z)$ for all $z \in D$, and so $\tau \approx^y_x \sigma$ by **D25**. $\square$

L18 *$\mathcal{M}, \tau, x \vDash \varphi$ just in case $\mathcal{M}, \sigma, y \vDash \varphi$ for any model $\mathcal{M} = \langle W, \mathcal{D}, \Rightarrow, |\cdot|\rangle$ of $\mathcal{L}$, well-formed sentence φ of $\mathcal{L}$, and possible worlds $\tau, \sigma \in H_{\mathcal{F}}$ where $\tau \approx^y_x \sigma$.*

Proof. We proceed by induction on the complexity of φ, where $\tau \approx^y_x \sigma$ yields the identity $\tau(z) = \sigma(z + y - x)$ for every $z \in D$ by **D25**. For a sentence letter p_i, taking $z = x$ gives $\tau(x) = \sigma(y)$, and so $\mathcal{M}, \tau, x \vDash p_i$ iff $\tau(x) \in |p_i|$ iff $\sigma(y) \in |p_i|$ iff $\mathcal{M}, \sigma, y \vDash p_i$ by **D24**. The case of $\bot$ is immediate since neither side holds, and the case of $\varphi \to \psi$ follows from the inductive hypothesis for φ and ψ through the ($\to$) clause.

For $\Box\varphi$, suppose $\mathcal{M}, \tau, x \nvDash \Box\varphi$, so that $\mathcal{M}, \rho, x \nvDash \varphi$ for some $\rho \in H_{\mathcal{F}}$. By **L17** there is some $\rho' \in H_{\mathcal{F}}$ where $\rho \approx_x^y \rho'$, whence $\mathcal{M}, \rho', y \nvDash \varphi$ by the inductive hypothesis, and so $\mathcal{M}, \sigma, y \nvDash \Box\varphi$. The converse is symmetric.

For $\varphi \lhd \psi$, translation by $y - x$ preserves the order on D and carries each $z \in D$ to $z' \coloneqq z + y - x$ where $\tau \approx_z^{z'} \sigma$ by **D25**, so the inductive hypothesis applies at every pair of corresponding times. Thus if $\mathcal{M}, \tau, x \vDash \varphi \lhd \psi$ with witness $z < x$ where $\mathcal{M}, \tau, z \vDash \psi$ and $\mathcal{M}, \tau, w \vDash \varphi$ for all $w \in D$ with $z < w < x$, then $z' < y$ where $\mathcal{M}, \sigma, z' \vDash \psi$ and $\mathcal{M}, \sigma, w' \vDash \varphi$ for all $w' \in D$ with $z' < w' < y$, every such w' being the translate of some w with $z < w < x$, and so $\mathcal{M}, \sigma, y \vDash \varphi \lhd \psi$. The converse translates witnesses by $x - y$ in the same way, and the case of $\varphi \rhd \psi$ is symmetric. $\square$

D26 A conclusion φ is a *logical consequence* of a set of premises Γ— written $\Gamma \vDash \varphi$— just in case for all models $\mathcal{M}$, possible worlds $\tau \in H_{\mathcal{F}}$, and times $x \in D$, if $\mathcal{M}, \tau, x \vDash \gamma$ for all premises $\gamma \in \Gamma$, then $\mathcal{M}, \tau, x \vDash \varphi$. A sentence φ is *valid* just in case $\vDash \varphi$.

D27 A well-formed sentence φ of $\mathcal{L}$ is *valid over a task frame* $\mathcal{F} = \langle W, \mathcal{D}, \Rightarrow\rangle$ which we may write $\vDash_{\mathcal{F}} \varphi$ if and only if $\mathcal{M}, \tau, x \vDash \varphi$ for every model $\mathcal{M} = \langle W, \mathcal{D}, \Rightarrow, |\cdot|\rangle$ where $\mathcal{F} = \langle W, \mathcal{D}, \Rightarrow\rangle$, possible world $\tau \in H_{\mathcal{F}}$, and time $x \in D$.[87]

D28 For any class C of task frames, a well-formed sentence φ is *valid over* C— written $\vDash_{\mathsf{C}} \varphi$— just in case $\vDash_{\mathcal{F}} \varphi$ for every $\mathcal{F} \in \mathsf{C}$, and $\Gamma \vDash_{\mathsf{C}} \varphi$ restricts **D26** to models over task frames in C. For any temporal order $\mathcal{D}$, the *fibre* $\mathsf{T}_{\mathcal{D}}$ is the class of all task frames $\langle W, \mathcal{D}, \Rightarrow\rangle$ with temporal order $\mathcal{D}$, where $\vDash_{\mathcal{D}} \varphi$ abbreviates $\vDash_{\mathsf{T}_{\mathcal{D}}} \varphi$. Thus $\vDash \varphi$ is validity over the class of all task frames, and $\vDash_{\mathcal{F}} \varphi$ is validity over $\{\mathcal{F}\}$.

D29 A temporal order $\mathcal{D} = \langle D, +, 0, \leqslant\rangle$ is:

Discrete if for any $x \in D$, whenever there exists $y > x$, there is a least such $y' > x$ satisfying $z \geqslant y'$ for all $z > x$.

Dense if for any $x, y \in D$ where $x < y$, there exists $z \in D$ where $x < z < y$.

Complete if every nonempty $S \subseteq D$ bounded above has a least upper bound in D.

A task frame $\mathcal{F} = \langle W, \mathcal{D}, \Rightarrow\rangle$ inherits these properties from $\mathcal{D}$.

T10 $\vDash_{\mathcal{D}} (\boxed{\scriptstyle P}\varphi \wedge \varphi \wedge \Diamond_{F}\top) \to \Diamond_{F}\boxed{\scriptstyle P}\varphi$ *iff* $\mathcal{D}$ *is* Discrete.[88]

Proof. ($\Leftarrow$) Suppose $\mathcal{D}$ is Discrete and let $\mathcal{F} = \langle W, \mathcal{D}, \Rightarrow\rangle$ be a task frame. Let $\mathcal{M} = \langle W, \mathcal{D}, \Rightarrow, |\cdot|\rangle$ be a model with $\tau \in H_{\mathcal{F}}$ and $x \in D$ where $\mathcal{M}, \tau, x \vDash \boxed{\scriptstyle P}\varphi \wedge \varphi \wedge \Diamond_{F}\top$.

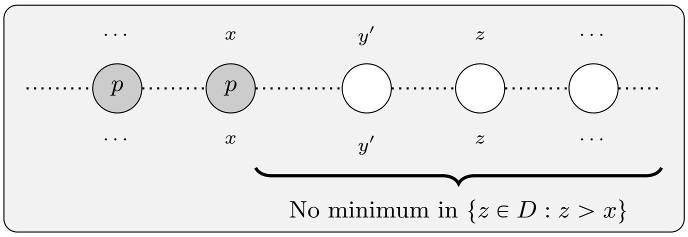
[87] Since $H_{\mathcal{F}} \neq \varnothing$ for every task frame by **C3**, frame validity is never vacuous: every task frame contributes evaluation points, and so $\nvDash_{\mathcal{F}} \bot$ for every task frame $\mathcal{F}$.

[88] The theorems of this kind cannot be sharpened to single task frames. The *static* task frame over $\mathcal{D}$, in which $w \Rightarrow_x u$ just in case $w = u$, satisfies every clause of **D19**— each cone and each nonempty fiber and segment being a singleton— yet its possible worlds are constant, so that every sentence of $\mathcal{L}$ has the same truth value at every time along a possible world and **DF**, **DN**, and **CO** are all valid over it whatever $\mathcal{D}$ may be. Since the tense operators see $\mathcal{D}$ only through the convex histories that $\Rightarrow$ admits, correspondence holds over the fibre $\mathsf{T}_{\mathcal{D}}$ rather than frame by frame, as the Lean 4 repository for this paper records.

Since $\Diamond\top$ provides some $y > x$, there is a least $x^+ > x$ by **D29**. Every $t < x^+$ then satisfies $t \leqslant x$, where $\mathcal{M}, \tau, t \vDash \varphi$ by $\boxed{\mathrm{P}}\varphi$ when $t < x$ and by φ itself when $t = x$. Thus $\mathcal{M}, \tau, x^+ \vDash \boxed{\mathrm{P}}\varphi$, giving $\mathcal{M}, \tau, x \vDash \Diamond\boxed{\mathrm{P}}\varphi$. Since $\mathcal{F}$, $\mathcal{M}$, τ, and x were arbitrary, $(\boxed{\mathrm{P}}\varphi \wedge \varphi \wedge \Diamond\top) \to \Diamond\boxed{\mathrm{P}}\varphi$ is valid over every task frame $\mathcal{F} = \langle W, \mathcal{D}, \Rightarrow\rangle$.

($\Rightarrow$) Suppose $\mathcal{D}$ is not Discrete. Then there exists $x \in D$ such that $\{z \in D : z > x\}$ is nonempty but has no minimum. Let $W = D$ with $w \Rightarrow_d u$ *iff* $u = w + d$ for $d \geqslant 0$, let $\mathcal{F}' = \langle W, \mathcal{D}, \Rightarrow\rangle$, and define $\mathcal{M} = \langle W, \mathcal{D}, \Rightarrow, |\cdot|\rangle$ with $|p| = \{w \in W : w \leqslant x\}$ and $\tau : D \to W$ by $\tau(y) = y$. Both directions of *Compositionality* are immediate with $u = w + x$ the unique intermediate state for $w \Rightarrow_{x+y} v$. *Seriality* holds since the forward fiber of w at d is $\{w + d\}$ and the backward fiber is $\{w - d\}$, and since $(w)_d = \{u \in W : |u - w| < d\}$, we have $\bigcap_{d>0}(w)_d = \{w\}$ for *Limit*. Since every fiber and nonempty segment is a singleton, so any two members of a $\supseteq$-directed family of nonempty fibers and segments include a common singleton member and so coincide, leaving one singleton whose intersection is nonempty, and so *Saturation* holds. The possible worlds over $\mathcal{F}'$ are the translations $\sigma_c(z) = z + c$. In particular, $\tau = \sigma_0 \in H_{\mathcal{F}'}$.

τ:



Thus $\mathcal{M}, \tau, x \vDash p$ since $\tau(x) = x \in |p|$, and $\mathcal{M}, \tau, x \vDash \boxed{\mathrm{P}}p$ since $\tau(y) = y \in |p|$ for all $y < x$ by **D24**. Since $\{z \in D : z > x\}$ is nonempty, $\mathcal{M}, \tau, x \vDash \Diamond\top$. Suppose for contradiction that $\mathcal{M}, \tau, x \vDash \Diamond\boxed{\mathrm{P}}p$. Then there exists $z > x$ with $\mathcal{M}, \tau, z \vDash \boxed{\mathrm{P}}p$. Since there is no minimum in $\{z' \in D : z' > x\}$, there exists $y' \in D$ with $x < y' < z$. But $\tau(y') = y' \notin |p|$ since $y' > x$, so $\mathcal{M}, \tau, y' \nvDash p$ by **D24**, contradicting $\mathcal{M}, \tau, z \vDash \boxed{\mathrm{P}}p$. Therefore $\mathcal{M}, \tau, x \nvDash \Diamond\boxed{\mathrm{P}}p$, and so $\nvDash_{\mathcal{F}'} (\boxed{\mathrm{P}}\varphi \wedge \varphi \wedge \Diamond\top) \to \Diamond\boxed{\mathrm{P}}\varphi$, and the axiom is not valid over every task frame with temporal order $\mathcal{D}$. $\square$

P3 *For any* Discrete *temporal order $\mathcal{D}$ that is not Archimedean, we show that both $\nvDash_{\mathcal{D}} \Diamond\varphi \to (\neg\varphi \rhd \varphi)$ and $\nvDash_{\mathcal{D}} \boxed{\mathrm{F}}(\boxed{\mathrm{F}}\varphi \to \varphi) \to (\Diamond\boxed{\mathrm{F}}\varphi \to \boxed{\mathrm{F}}\varphi)$.*

Proof. Since $\mathcal{D}$ is nontrivial and Discrete, there is a least duration $1 > 0$, and since $\mathcal{D}$ is not Archimedean, there is some $y > 0$ where $y > n \cdot 1$ for every positive integer n, as in $\mathbb{Z} \times_{\mathrm{lex}} \mathbb{Z}$, where $1 = \langle 0, 1\rangle$ and y may be taken to be $\langle 1, 0\rangle$. Let $\mathcal{F}' = \langle W, \mathcal{D}, \Rightarrow\rangle$ be the translation task frame from the proof of **T10**, where $W = D$ and $w \Rightarrow_d u$ just in case $u = w + d$ for $d \geqslant 0$, whose possible worlds are the translations $\sigma_c(z) = z + c$, so that $\tau = \sigma_0 \in H_{\mathcal{F}'}$. Since $\mathcal{D}$ is an ordered group, the least time greater than any $z \in D$ is $z + 1$. In either case the countermodel turns on the same division of D: the times within finitely many steps of 0— those below some multiple $n \cdot 1$— and those beyond every such multiple, of which y is one.

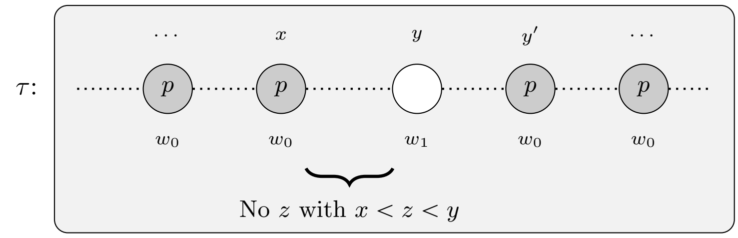
For **Z1**, let $E = \{z \in D : 0 < z < n \cdot 1 \text{ for some } n \in \mathbb{N}\}$ and $\mathcal{M} = \langle W, \mathcal{D}, \Rightarrow, |\cdot|\rangle$ with $|p| = D \backslash E$. Since y exceeds every $n \cdot 1$, so does every $z > y$, and so $z \notin E$ and $\mathcal{M}, \tau, z \vDash p$ for all $z > y$, giving $\mathcal{M}, \tau, y \vDash \boxed{\mathsf{F}}p$ and hence $\mathcal{M}, \tau, 0 \vDash \Diamond_{\mathsf{F}}\boxed{\mathsf{F}}p$ by **D24**. Let $z > 0$ be arbitrary. If $z \notin E$, then $\mathcal{M}, \tau, z \vDash p$, and so $\mathcal{M}, \tau, z \vDash \boxed{\mathsf{F}}p \to p$. If $z \in E$, then $0 < z < n \cdot 1$ for some n, so $0 < z+1 < (n+1) \cdot 1$ and $z+1 \in E$, whence $\mathcal{M}, \tau, z+1 \nvDash p$ and $\mathcal{M}, \tau, z \nvDash \boxed{\mathsf{F}}p$, and so $\mathcal{M}, \tau, z \vDash \boxed{\mathsf{F}}p \to p$. Thus $\mathcal{M}, \tau, 0 \vDash \boxed{\mathsf{F}}(\boxed{\mathsf{F}}p \to p)$. But $1 \in E$ since $0 < 1 < 2 \cdot 1$, so $\mathcal{M}, \tau, 1 \nvDash p$ and $\mathcal{M}, \tau, 0 \nvDash \boxed{\mathsf{F}}p$. Therefore it follows that $\mathcal{M}, \tau, 0 \nvDash \boxed{\mathsf{F}}(\boxed{\mathsf{F}}p \to p) \to (\Diamond_{\mathsf{F}}\boxed{\mathsf{F}}p \to \boxed{\mathsf{F}}p)$, and so $\nvDash_{\mathcal{D}} \boxed{\mathsf{F}}(\boxed{\mathsf{F}}\varphi \to \varphi) \to (\Diamond_{\mathsf{F}}\boxed{\mathsf{F}}\varphi \to \boxed{\mathsf{F}}\varphi)$.

For **UZ**, define $\mathcal{M}$ over the same $\mathcal{F}'$ with $|p| = \{y + n \cdot 1 : n \in \mathbb{Z}\}$ instead, a copy of $\mathbb{Z}$ lying wholly above 0 since y exceeds every multiple of 1. Since $y > 0$ and $\mathcal{M}, \tau, y \vDash p$, we have $\mathcal{M}, \tau, 0 \vDash \Diamond_{\mathsf{F}}p$. But each p-time $y + n \cdot 1$ is preceded by the p-time $y + (n-1) \cdot 1 > 0$, so no p-time $z > 0$ has $\mathcal{M}, \tau, v \vDash \neg p$ for all $0 < v < z$, and so $\mathcal{M}, \tau, 0 \nvDash \neg p \rhd p$ by **D24**. Therefore $\nvDash_{\mathcal{D}} \Diamond_{\mathsf{F}}\varphi \to (\neg\varphi \rhd \varphi)$. □

T11 $\vDash_{\mathcal{D}} \boxed{\mathsf{F}}\boxed{\mathsf{F}}\varphi \to \boxed{\mathsf{F}}\varphi$ *iff* $\mathcal{D}$ *is* DENSE.

Proof. ($\Leftarrow$) Suppose $\mathcal{D}$ is DENSE and let $\mathcal{F} = \langle W, \mathcal{D}, \Rightarrow\rangle$ be an arbitrary task frame. Let $\mathcal{M} = \langle W, \mathcal{D}, \Rightarrow, |\cdot|\rangle$ be a model with $\tau \in H_{\mathcal{F}}$ and $x \in D$ where $\mathcal{M}, \tau, x \vDash \boxed{\mathsf{F}}\boxed{\mathsf{F}}\varphi$, and consider any $y > x$. Since $\mathcal{D}$ is DENSE, there is some $z \in D$ where $x < z < y$ by **D29**, whence $\mathcal{M}, \tau, z \vDash \boxed{\mathsf{F}}\varphi$ by **D24** and so $\mathcal{M}, \tau, y \vDash \varphi$ since $y > z$. Since $y > x$ was arbitrary, $\mathcal{M}, \tau, x \vDash \boxed{\mathsf{F}}\varphi$, and since $\mathcal{F}$, $\mathcal{M}$, τ, and x were arbitrary, $\boxed{\mathsf{F}}\boxed{\mathsf{F}}\varphi \to \boxed{\mathsf{F}}\varphi$ is valid over every task frame $\mathcal{F} = \langle W, \mathcal{D}, \Rightarrow\rangle$.

($\Rightarrow$) Suppose $\mathcal{D}$ is not DENSE. Then there exist $x, y \in D$ with $x < y$ and no $z \in D$ satisfying $x < z < y$. Thus $\epsilon \coloneqq y - x$ is the least positive duration in D, since any positive $d < \epsilon$ would give $x < x + d < y$. Let $W = \{w_0, w_1\}$ where $w \Rightarrow_0 w'$ *iff* $w = w'$ and $w \Rightarrow_d w'$ for all $w, w' \in W$ and $d > 0$, let $\mathcal{F}' = \langle W, \mathcal{D}, \Rightarrow\rangle$, and define $\mathcal{M} = \langle W, \mathcal{D}, \Rightarrow, |\cdot|\rangle$ with $|p| = \{w_0\}$ and $\tau : D \to W$ by $\tau(y) = w_1$ and $\tau(z) = w_0$ for all $z \neq y$. Both directions of *Compositionality* are immediate, taking $u = w$ or $u = w'$ for the factoring direction when a summand is zero and any $u \in W$ otherwise, *Seriality* holds since every fiber at a positive duration is all of W while every state loops at duration zero, and $(w)_\epsilon = \{w\}$ gives $\bigcap_{d>0}(w)_d = \{w\}$ for *Limit*. *Saturation* holds by **L10** since W is finite. Since every non-zero difference between times is covered by the relation, every function from D to W is a possible world over $\mathcal{F}'$, and so $\tau \in H_{\mathcal{F}'}$.



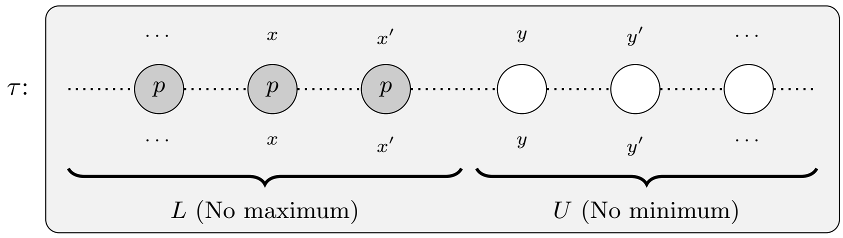
Given that $y > x$ and $\tau(y) = w_1 \notin |p|$, we have $\mathcal{M}, \tau, x \nvDash \boxed{\mathsf{F}}p$ by **D24**. Any $y' > x$ satisfies $y' \geqslant y$ since no time lies strictly between x and y, so every $z' > y'$ satisfies $z' > y$ and hence $\tau(z') = w_0 \in |p|$, giving $\mathcal{M}, \tau, y' \vDash \boxed{\mathsf{F}}p$. Thus $\mathcal{M}, \tau, x \vDash \boxed{\mathsf{F}}\boxed{\mathsf{F}}p$ by

D24, and so $\nvDash_{\mathcal{F}'} \boxed{\mathrm{F}}\boxed{\mathrm{F}}\varphi \to \boxed{\mathrm{F}}\varphi$, leaving the axiom invalid over some task frame with temporal order $\mathcal{D}$. $\square$

T12 $\quad \vDash_{\mathcal{D}} \triangle(\boxed{\mathrm{P}}\varphi \to \Diamond\boxed{\mathrm{P}}\varphi) \to (\boxed{\mathrm{P}}\varphi \to \boxed{\mathrm{F}}\varphi)$ *iff* $\mathcal{D}$ *is* Complete.

Proof. ($\Leftarrow$) Suppose $\mathcal{D}$ is Complete and let $\mathcal{F} = \langle W, \mathcal{D}, \Rightarrow\rangle$ be an arbitrary task frame. Let $\mathcal{M} = \langle W, \mathcal{D}, \Rightarrow, |\cdot|\rangle$ be a model with $\mathcal{M}, \tau, x \vDash \triangle(\boxed{\mathrm{P}}\varphi \to \Diamond\boxed{\mathrm{P}}\varphi)$ and $\mathcal{M}, \tau, x \vDash \boxed{\mathrm{P}}\varphi$ for $\tau \in H_{\mathcal{F}}$ and $x \in D$. Suppose for contradiction that $\mathcal{M}, \tau, x \nvDash \boxed{\mathrm{F}}\varphi$, so that $B := \{z \in D : z > x \text{ and } \mathcal{M}, \tau, z \nvDash \varphi\}$ is nonempty. Since B is bounded below by x, its reflection $-B$ is bounded above by $-x$ and so has a least upper bound by **D29**, whence $c := \inf(B)$ exists in D with $c \geqslant x$. Since $\mathcal{M}, \tau, x \vDash \boxed{\mathrm{P}}\varphi$ and $\mathcal{M}, \tau, x \vDash \boxed{\mathrm{P}}\varphi \to \Diamond\boxed{\mathrm{P}}\varphi$, there exists $x' > x$ with $\mathcal{M}, \tau, x' \vDash \boxed{\mathrm{P}}\varphi$, so no element of B is less than x' and hence $c \geqslant x' > x$. Thus $\mathcal{M}, \tau, c \vDash \boxed{\mathrm{P}}\varphi$, since every $y' < c$ either satisfies $y' < x'$, where φ holds by $\boxed{\mathrm{P}}\varphi$ at x', or $x < y' < c$, where $y' \notin B$. Given that $\mathcal{M}, \tau, c \vDash \boxed{\mathrm{P}}\varphi \to \Diamond\boxed{\mathrm{P}}\varphi$, there exists $z > c$ with $\mathcal{M}, \tau, z \vDash \boxed{\mathrm{P}}\varphi$, while $c = \inf(B)$ provides some $b \in B$ with $c \leqslant b < z$, where $\mathcal{M}, \tau, b \nvDash \varphi$ contradicts $\mathcal{M}, \tau, z \vDash \boxed{\mathrm{P}}\varphi$. Therefore $\mathcal{M}, \tau, x \vDash \boxed{\mathrm{F}}\varphi$. Since $\mathcal{F}$, $\mathcal{M}$, τ, and x were arbitrary, we may conclude that $\triangle(\boxed{\mathrm{P}}\varphi \to \Diamond\boxed{\mathrm{P}}\varphi) \to (\boxed{\mathrm{P}}\varphi \to \boxed{\mathrm{F}}\varphi)$ is valid over every task frame $\mathcal{F} = \langle W, \mathcal{D}, \Rightarrow\rangle$.

($\Rightarrow$) Suppose $\mathcal{D}$ is not Complete. Then there exists a nonempty $S \subseteq D$ bounded above with no least upper bound. Let $L = \{x \in D : x < y \text{ for some } y \in S\}$ and $U = \{x \in D : y \leqslant x \text{ for all } y \in S\}$. Since $\mathcal{D}$ is totally ordered, L and U partition D with every element of L strictly less than every element of U. Since S has no least upper bound, L has no maximum and U has no minimum. Let $W = D$ with $w \Rightarrow_d u$ *iff* $u = w + d$ for $d \geqslant 0$, let $\mathcal{F}' = \langle W, \mathcal{D}, \Rightarrow\rangle$, and define $\mathcal{M} = \langle W, \mathcal{D}, \Rightarrow, |\cdot|\rangle$ with $|p| = L$ and $\tau : D \to W$ by $\tau(x) = x$. As in **T10**, the task frame $\mathcal{F}'$ satisfies *Compositionality*, *Seriality*, *Limit*, and *Saturation*, where $\tau \in H_{\mathcal{F}'}$ is total.



Given **D24**, it follows that $\mathcal{M}, \tau, x \vDash p$ if and only if $x \in L$. Pick any $x \in L$. Since L is downward closed, $\mathcal{M}, \tau, x \vDash \boxed{\mathrm{P}}p$. Since U is nonempty and every $y \in U$ satisfies $y > x$ and $\tau(y) = y \notin |p|$, we have $\mathcal{M}, \tau, x \nvDash \boxed{\mathrm{F}}p$.

It remains to verify $\mathcal{M}, \tau, x \vDash \triangle(\boxed{\mathrm{P}}p \to \Diamond\boxed{\mathrm{P}}p)$. Let $y \in D$ be arbitrary. If $y \in U$, then since U has no minimum, there exists $y' \in U$ with $y' < y$, and $\tau(y') = y' \notin |p|$, so $\mathcal{M}, \tau, y \nvDash \boxed{\mathrm{P}}p$ and the conditional holds vacuously. If $y \in L$, then $\mathcal{M}, \tau, y \vDash \boxed{\mathrm{P}}p$ since L is downward closed. Since L has no maximum, there exists $y' \in L$ with $y' > y$. Every $z < y'$ lies in L by downward closure, so $\mathcal{M}, \tau, y' \vDash \boxed{\mathrm{P}}p$, giving $\mathcal{M}, \tau, y \vDash \Diamond\boxed{\mathrm{P}}p$. Since y was arbitrary, $\mathcal{M}, \tau, x \vDash \triangle(\boxed{\mathrm{P}}p \to \Diamond\boxed{\mathrm{P}}p)$. Since $\mathcal{M}, \tau, x \vDash \boxed{\mathrm{P}}p$ while $\mathcal{M}, \tau, x \nvDash \boxed{\mathrm{F}}p$, we have $\not\models_{\mathcal{F}'} \triangle(\boxed{\text{P}}\varphi \to \Diamond_{\text{F}}\boxed{\text{P}}\varphi) \to (\boxed{\text{P}}\varphi \to \boxed{\text{F}}\varphi)$, leaving the axiom invalid over some task frame with temporal order $\mathcal{D}$. □

D30 A task frame $\mathcal{F} = \langle W, \mathcal{D}, \Rightarrow\rangle$ is Deterministic just in case $u = v$ whenever $w \Rightarrow_x u$ and $w \Rightarrow_x v$ for $w, u, v \in W$ and $x \in D$, holding in both temporal directions since **D18**'s reflection convention already extends x over all of D.

D31 For a task frame $\mathcal{F} = \langle W, \mathcal{D}, \Rightarrow\rangle$, possible world $\tau \in H_{\mathcal{F}}$, and time $x \in D$, let $\langle\tau\rangle_x \coloneqq \{\sigma \in H_{\mathcal{F}} \mid \sigma(x) = \tau(x)\}$ be the set of possible worlds that intersect τ at x. Adding a vector $\vec{v} = \langle v_1, v_2, \ldots\rangle$ of stored times to the point of evaluation, the clauses of **D24** are unchanged and $\vec{v}$ may be omitted when no store or recall operator occurs. Since $\uparrow^i_{\text{M}}$ and $\downarrow^i_{\text{M}}$ do not occur below, the vector $\vec{\mu}$ of stored worlds from §**3.3** may likewise be suppressed throughout, where the remaining operators are interpreted by:

($\boxdot$) $\mathcal{M}, \tau, x, \vec{v} \models \boxdot\varphi$ *iff* $\mathcal{M}, \sigma, x, \vec{v} \models \varphi$ for all $\sigma \in \langle\tau\rangle_x$.

($\uparrow_{\text{T}}$) $\mathcal{M}, \tau, x, \vec{v} \models \uparrow^i_{\text{T}}\varphi$ *iff* $\mathcal{M}, \tau, x, \vec{v}_{[x/v_i]} \models \varphi$.

($\downarrow_{\text{T}}$) $\mathcal{M}, \tau, x, \vec{v} \models \downarrow^i_{\text{T}}\varphi$ *iff* $\mathcal{M}, \tau, v_i, \vec{v} \models \varphi$.

L19 *$\mathcal{F}$ is* Deterministic *if and only if $\langle\tau\rangle_x = \{\tau\}$ for all $\tau \in H_{\mathcal{F}}$ and $x \in D$.*[89]

Proof. Suppose $\mathcal{F}$ is Deterministic. Since $\tau(x) = \tau(x)$, we have $\tau \in \langle\tau\rangle_x$. For the reverse inclusion, let $\sigma \in \langle\tau\rangle_x$ so that $\sigma(x) = \tau(x)$. For any $y \in D$, since τ is a possible world, $\tau(x) \Rightarrow_{y-x} \tau(y)$ by **D21**. Since σ is a possible world, $\sigma(x) \Rightarrow_{y-x} \sigma(y)$ by **D21**, and so $\tau(x) \Rightarrow_{y-x} \sigma(y)$. By Deterministic (**D30**), $\tau(y) = \sigma(y)$. Since y was arbitrary, we may conclude that $\sigma = \tau$ as desired.

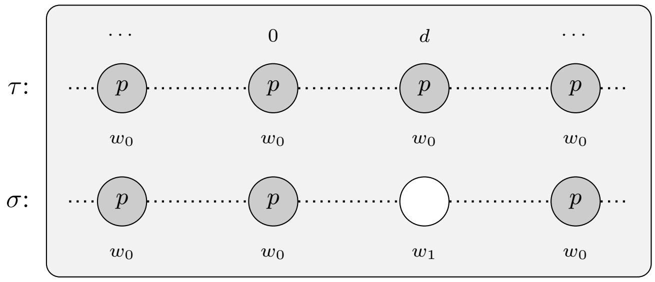
For the converse, suppose $\mathcal{F}$ is not Deterministic, so that $w \Rightarrow_x u$ and $w \Rightarrow_x v$ for some $w, u, v \in W$ and $x \in D$ where $u \neq v$. Observe first that $x \neq 0$. For if $w \Rightarrow_0 u'$, then $u' \in \texttt{fib}(w, 0) \subseteq (w)_y$ for every $y > 0$ since $|0| < y$, and so $u' \in \bigcap_{y>0}(w)_y = \{w\}$ by *Limit*, giving $u' = w$; together with **L9** this makes $\Rightarrow_0$ the identity on W, which admits no such failure. Consider then the two-point functions $\tau_1, \tau_2 : \{0, x\} \to W$ where $\tau_1(0) = \tau_2(0) = w$ while $\tau_1(x) = u$ and $\tau_2(x) = v$. Each is a partial history by **D21**: the instances at equal times are $w \Rightarrow_0 w$, $u \Rightarrow_0 u$, and $v \Rightarrow_0 v$, all given by **L9**; the instance from 0 to x is the hypothesis; and the instance from x to 0 is $u \Rightarrow_{-x} w$, which is that same hypothesis under **D18**'s reflection convention. Since convexity of the domain is required only of convex histories whereas **T9** is stated for partial histories, the non-convex domain $\{0, x\}$ is admissible, and so **T9** yields possible worlds $T_1, T_2 \in H_{\mathcal{F}}$ extending τ_1 and τ_2 respectively. Given that $T_1(0) = w = T_2(0)$, we have $T_2 \in \langle T_1\rangle_0$, while $T_1(x) = u \neq v = T_2(x)$ gives $T_1 \neq T_2$. Thus $\langle T_1\rangle_0 \neq \{T_1\}$, and the contrapositive establishes the converse. □

T13 $\models_{\mathcal{F}} \varphi \to \boxdot\varphi$ *for every* Deterministic *task frame $\mathcal{F}$, while $\not\models_{\mathcal{F}'} \varphi \to \boxdot\varphi$ for some task frame $\mathcal{F}'$ that is not* Deterministic.

[89] Only the left-to-right direction is choice-free. The converse appeals to **T9** and hence to Zorn's lemma, and so is a theorem of ZFC, as with **C3**.

Proof. Let $\mathcal{F} = \langle W, \mathcal{D}, \Rightarrow \rangle$ be DETERMINISTIC. Let $\mathcal{M} = \langle W, \mathcal{D}, \Rightarrow, |\cdot| \rangle$ be a model over $\mathcal{F}$ with $\tau \in H_{\mathcal{F}}$ and $x \in D$ where $\mathcal{M}, \tau, x \vDash \varphi$. By **L19**, $\langle \tau \rangle_x = \{\tau\}$, so $\mathcal{M}, \sigma, x \vDash \varphi$ for all $\sigma \in \langle \tau \rangle_x$. Therefore $\mathcal{M}, \tau, x \vDash \boxdot\varphi$ by **D31**. Since $\mathcal{M}$, τ, and x were chosen arbitrarily, $\vDash_{\mathcal{F}} \varphi \to \boxdot\varphi$ for every deterministic task frame $\mathcal{F}$.

Let $\mathcal{D}$ be a discrete temporal order with least positive duration d. Given the set $W = \{w_0, w_1\}$ where $w \Rightarrow_0 w'$ *iff* $w = w'$ and $w \Rightarrow_z w'$ for all $w, w' \in W$ and $z > 0$, let $\mathcal{F}' = \langle W, \mathcal{D}, \Rightarrow \rangle$, and define $\mathcal{M} = \langle W, \mathcal{D}, \Rightarrow, |\cdot| \rangle$ with $|p| = \{w_0\}$ and $\tau, \sigma : D \to W$ by $\tau(y) = w_0$ for all $y \in D$, and $\sigma(y) = w_1$ if $y = d$ and $\sigma(y) = w_0$ otherwise. This is the two-state frame of **T11**, verified there to satisfy *Compositionality*, *Seriality*, *Limit*, and *Saturation*, and to admit every function from D to W as a possible world, so that $\tau, \sigma \in H_{\mathcal{F}'}$. Since $w_0 \Rightarrow_d w_0$ and $w_0 \Rightarrow_d w_1$, the task frame $\mathcal{F}'$ is not DETERMINISTIC.



Since $\sigma(0) = w_0 = \tau(0)$, we have $\sigma \in \langle \tau \rangle_0$, while **D24** gives $\mathcal{M}, \tau, 0 \vDash \boxed{\scriptstyle \mathrm{F}}p$ since $\tau(y) = w_0 \in |p|$ for all $y > 0$, and $\mathcal{M}, \sigma, 0 \nvDash \boxed{\scriptstyle \mathrm{F}}p$ since $\sigma(d) = w_1 \notin |p|$ where $d > 0$. Thus σ witnesses $\mathcal{M}, \tau, 0 \nvDash \boxdot\boxed{\scriptstyle \mathrm{F}}p$ by **D31**, and so $\nvDash_{\mathcal{F}'} \varphi \to \boxdot\varphi$ over a task frame that is not DETERMINISTIC. $\square$

D32 Let W be linearly ordered by $<$ where $|p_i| \subseteq W$ for every sentence letter $p_i \in \mathbb{L}$. The *state set* $|\varphi| \subseteq W$ of each sentence φ of $\mathcal{L}$ extended with $\boxdot$ extends that assignment by the recursion below, which appeals to just $<$ and the sets $|p_i|$:

($\bot$) $|\bot| \coloneqq \varnothing$.

($\to$) $|\varphi \to \psi| \coloneqq (W \backslash |\varphi|) \cup |\psi|$.

($\Box$) $|\Box\varphi| \coloneqq W$ if $|\varphi| = W$, and $|\Box\varphi| \coloneqq \varnothing$ otherwise.

($\lhd$) $|\varphi \lhd \psi| \coloneqq \{w \in W \mid v \in |\psi| \text{ and } (v, w) \subseteq |\varphi| \text{ for some } v < w\}$.

($\rhd$) $|\varphi \rhd \psi| \coloneqq \{w \in W \mid v \in |\psi| \text{ and } (w, v) \subseteq |\varphi| \text{ for some } v > w\}$.

($\boxdot$) $|\boxdot\varphi| \coloneqq |\varphi|$.

Since these clauses mention no task frame, and so neither its possible worlds nor its task relation, any two task frames over the same linearly ordered W assign the same state sets by definition rather than by proof, as **C4** requires.

D33 A task frame $\mathcal{F} = \langle W, \mathcal{D}, \Rightarrow \rangle$ satisfies Flow just in case W is linearly ordered by $<$ and every possible world $\tau \in H_{\mathcal{F}}$ is an order-isomorphism of $\langle D, < \rangle$ onto $\langle W, < \rangle$.

L20 *Let $\mathcal{F} = \langle W, \mathcal{D}, \Rightarrow \rangle$ be a task frame satisfying* Flow *as in* ***D33****, so that W is linearly ordered by $<$. For every model $\mathcal{M} = \langle W, \mathcal{D}, \Rightarrow, |\cdot| \rangle$ over $\mathcal{F}$, sentence φ of $\mathcal{L}$ extended with $\boxdot$, possible world $\tau \in H_{\mathcal{F}}$, and time $x \in D$, we have $\mathcal{M}, \tau, x \vDash \varphi$ iff $\tau(x) \in |\varphi|$, where $|\varphi|$ is the state set that* ***D32*** *generates from the assignment $|\cdot|$ of $\mathcal{M}$ to the sentence letters. Accordingly $\vDash_{\mathcal{F}} \varphi \to \boxdot\varphi$, while $\vDash_{\mathcal{F}} \varphi$ just in case $|\varphi| = W$ under every such assignment.*

Proof. The proof goes by induction on complexity, proving the biconditional for all $\tau \in H_{\mathcal{F}}$ and $x \in D$ at once, where **D24** gives the cases of p_i and $\perp$ outright and that of $\varphi \to \psi$ from the induction hypothesis. For $\varphi \rhd \psi$, the order-isomorphism τ carries the times $z > x$ onto the states $v > \tau(x)$ and each interval (x, z) onto $(\tau(x), \tau(z))$, so by the induction hypothesis $\mathcal{M}, \tau, x \vDash \varphi \rhd \psi$ just in case $v \in |\psi|$ and $(\tau(x), v) \subseteq |\varphi|$ for some $v > \tau(x)$, which is $\tau(x) \in |\varphi \rhd \psi|$ by **D32**, where $\varphi \lhd \psi$ is symmetric. For $\Box\varphi$, every world state occurs at every time by **C3**, so that $\{\rho(x) : \rho \in H_{\mathcal{F}}\} = W$ and the induction hypothesis makes $\mathcal{M}, \tau, x \vDash \Box\varphi$ equivalent to $|\varphi| = W$. For $\boxdot\varphi$, every $\sigma \in \langle \tau \rangle_x$ satisfies $\sigma(x) = \tau(x)$ by **D31**, so the induction hypothesis makes $\mathcal{M}, \sigma, x \vDash \varphi$ equivalent to $\tau(x) \in |\varphi|$ for each such σ, of which τ is one.

Since $|\boxdot\varphi| = |\varphi|$, the biconditional makes φ and $\boxdot\varphi$ true at exactly the same points, and so $\vDash_{\mathcal{F}} \varphi \to \boxdot\varphi$ by **D27**. Finally, if $|\varphi| = W$ under every assignment then $\tau(x) \in |\varphi|$ at every point of every model over $\mathcal{F}$, so $\vDash_{\mathcal{F}} \varphi$, while if $\vDash_{\mathcal{F}} \varphi$ then under any assignment each $w \in W$ is $\tau(x)$ for some $\tau \in H_{\mathcal{F}}$ by **C3**, whence $w \in |\varphi|$. □

T14 *There is a task frame that is not* Deterministic *over which $\varphi \to \boxdot\varphi$ is valid for every sentence φ of $\mathcal{L}$ extended with $\boxdot$.*

Proof. Let $\mathcal{F}^\circ = \langle W, \mathcal{D}, \Rightarrow \rangle$ where $W = \mathbb{R}$, $\mathcal{D} = \langle \mathbb{R}, +, 0, \leqslant \rangle$, and $w \Rightarrow_x u$ just in case $x \leqslant u - w \leqslant 2x$ for $x \geqslant 0$, extended to negative durations by **D18**, so that each world state drifts rightward along $\mathbb{R}$ at a rate nondeterministically between 1 and 2. Every fiber is then a compact interval, $\mathtt{fib}(w, x)$ being $[w + x, w + 2x]$ for $x \geqslant 0$ and $[w + 2x, w + x]$ for $x < 0$. For *Compositionality* with $x, y \geqslant 0$, if $w \Rightarrow_x u$ and $u \Rightarrow_y v$ then $v - w = (u - w) + (v - u)$ is a sum of terms drawn from $[x, 2x]$ and $[y, 2y]$ and so lies in $[x + y, 2(x + y)]$. Conversely, if $w \Rightarrow_{x+y} v$ with $x + y > 0$ then $u := w + \lambda x$ for $\lambda := (v - w)/(x + y) \in [1, 2]$ gives $u - w = \lambda x \in [x, 2x]$ and $v - u = \lambda y \in [y, 2y]$, while $x + y = 0$ forces $v = w$, where $u := w$ suffices. *Seriality* holds since $w \Rightarrow_x (w + x)$ and $(w - x) \Rightarrow_x w$ for every $w \in W$ and $x \geqslant 0$, and *Limit* since $w \in \mathtt{fib}(w, 0) \subseteq (w)_x$ while $(w)_x \subseteq (w - 2x, w + 2x)$ for every $x > 0$, whence $\bigcap_{x>0}(w)_x = \{w\}$. For *Saturation*, every segment is an intersection of two fibers and so is compact along with the fibers themselves, and a $\supseteq$-directed family of nonempty compact subsets of $\mathbb{R}$ has the finite intersection property and hence nonempty intersection. Thus $\mathcal{F}^\circ$ is a task frame, and it is not Deterministic by **D30** since $0 \Rightarrow_1 1$ and $0 \Rightarrow_1 2$.

By **D21**, the possible worlds in $H_{\mathcal{F}^\circ}$ are exactly the functions $\tau : \mathbb{R} \to \mathbb{R}$ where $y - x \leqslant \tau(y) - \tau(x) \leqslant 2(y - x)$ whenever $x < y$. Each is therefore strictly increasing and 2-Lipschitz hence continuous, and is unbounded in both directions since $\tau(t) \geqslant \tau(0) + t$

for $t > 0$ and $\tau(t) \leqslant \tau(0) + t$ for $t < 0$, and so by the intermediate value theorem is an order-isomorphism of $\langle \mathbb{R}, < \rangle$ onto itself. Thus $\mathcal{F}^\circ$ satisfies Flow, and so $\vDash_{\mathcal{F}^\circ} \varphi \to \boxdot\varphi$ by **L20**, whose appeals to **C3** the translation $\delta(t) \coloneqq t + w - x$ discharges directly, being a possible world with $\delta(x) = w$ for any $w \in W$ and $x \in D$.

In a model $\mathcal{M}$ over $\mathcal{F}^\circ$ with $|p| = [3/2, \infty)$, the possible worlds $\tau(t) \coloneqq t$ and $\sigma(t) \coloneqq 2t$ satisfy $\tau \in \langle \sigma \rangle_0$ while $\sigma(1) = 2 \in |p|$ and $\tau(1) = 1 \notin |p|$. Were the language to include store and recall operators, $\mathcal{M}, \sigma, 0, \vec{v}_{[1/v_2]} \vDash \downarrow^2_{\mathrm{T}} p$ but $\mathcal{M}, \sigma, 0, \vec{v}_{[1/v_2]} \nvDash \boxdot\downarrow^2_{\mathrm{T}} p$, witnessed by τ, refuting this instance of *Determined* over $\mathcal{F}^\circ$. Since σ witnesses the failure of $\boxdot\downarrow^2_{\mathrm{T}} \neg p$ at the same point, the disjunction in `Det` with p for φ fails at σ and 0 once 0 and 1 are stored as v_1 and v_2, although `Det` evaluated at σ and 0 requires it to hold for every future time so stored, the time 1 included. Thus $\mathcal{F}^\circ$ refutes `Det` while validating every instance of *Determined* without store and recall operators. $\square$

C4 *No set of sentences of $\mathcal{L}$ with $\boxdot$ defines the* Deterministic *task frames.*

Proof. Let $\mathcal{F}^1 = \langle W, \mathcal{D}, \Rightarrow^1 \rangle$ share $W = \mathbb{R}$ and $\mathcal{D}$ with the drift frame $\mathcal{F}^\circ$ of **T14**, but let $w \Rightarrow^1_x u$ just in case $u = w + x$, so that each world state drifts rightward at the constant rate 1. Every fiber $\mathtt{fib}(w, x)$ is then the singleton $\{w + x\}$, and so $\mathcal{F}^1$ is Deterministic by **D30**, while the verifications of **T14** go through with the interval $[x, 2x]$ narrowed to $\{x\}$ and the rate λ fixed at 1, making $\mathcal{F}^1$ a task frame. By **D21**, its possible worlds are exactly the translations $\tau(t) \coloneqq t + c$ for $c \in \mathbb{R}$, each an order-isomorphism of $\langle \mathbb{R}, < \rangle$ onto itself, and so $\mathcal{F}^1$ satisfies Flow just as $\mathcal{F}^\circ$ does. Thus **L20** applies to both frames over the same linearly ordered W: a sentence φ of $\mathcal{L}$ extended with $\boxdot$ is valid over $\mathcal{F}^\circ$, or over $\mathcal{F}^1$, just in case $|\varphi| = W$ under every assignment to the sentence letters, and **D32** builds $|\varphi|$ from the order on W and that assignment alone, never consulting the task relation. Exactly the same such sentences are therefore valid over $\mathcal{F}^\circ$ and $\mathcal{F}^1$. Were a set of them to define the Deterministic task frames, it would be valid over $\mathcal{F}^1$, hence over $\mathcal{F}^\circ$, which is not Deterministic. $\square$

T15 $\uparrow^1_{\mathrm{T}} \boxed{\scriptstyle F} \uparrow^2_{\mathrm{T}} \downarrow^1_{\mathrm{T}} (\boxdot\downarrow^2_{\mathrm{T}} \neg\varphi \vee \boxdot\downarrow^2_{\mathrm{T}} \varphi)$ *is valid over every deterministic task frame, and invalid over some non-deterministic task frame.*

Proof. We begin by observing that the biconditionals below follow from **D24** and **D31**:

$$\begin{aligned} &\mathcal{M}, \tau, x, \vec{v} \vDash \uparrow^1_{\mathrm{T}} \boxed{\scriptstyle F} \uparrow^2_{\mathrm{T}} \downarrow^1_{\mathrm{T}} (\boxdot\downarrow^2_{\mathrm{T}} \neg\varphi \vee \boxdot\downarrow^2_{\mathrm{T}} \varphi) && (*) \\ &\Leftrightarrow\ \mathcal{M}, \tau, x, \vec{v}_{[x/v_1]} \vDash \boxed{\scriptstyle F} \uparrow^2_{\mathrm{T}} \downarrow^1_{\mathrm{T}} (\boxdot\downarrow^2_{\mathrm{T}} \neg\varphi \vee \boxdot\downarrow^2_{\mathrm{T}} \varphi) \\ &\Leftrightarrow\ \mathcal{M}, \tau, y, \vec{v}_{[x/v_1]} \vDash \uparrow^2_{\mathrm{T}} \downarrow^1_{\mathrm{T}} (\boxdot\downarrow^2_{\mathrm{T}} \neg\varphi \vee \boxdot\downarrow^2_{\mathrm{T}} \varphi) \text{ for all } y > x \\ &\Leftrightarrow\ \mathcal{M}, \tau, y, \vec{v}_{[x/v_1][y/v_2]} \vDash \downarrow^1_{\mathrm{T}} (\boxdot\downarrow^2_{\mathrm{T}} \neg\varphi \vee \boxdot\downarrow^2_{\mathrm{T}} \varphi) \text{ for all } y > x \\ &\Leftrightarrow\ \mathcal{M}, \tau, x, \vec{v}_{[x/v_1][y/v_2]} \vDash \boxdot\downarrow^2_{\mathrm{T}} \neg\varphi \vee \boxdot\downarrow^2_{\mathrm{T}} \varphi \text{ for all } y > x. \end{aligned}$$

Let $\mathcal{F} = \langle W, \mathcal{D}, \Rightarrow \rangle$ be Deterministic. Choose an world $\tau \in H_{\mathcal{F}}$, times $x, y \in D$ where $x < y$, and stored times $\vec{v}$ in a model $\mathcal{M} = \langle W, \mathcal{D}, \Rightarrow, |\cdot| \rangle$ over $\mathcal{F}$. By **L19**, it follows that $\langle \tau \rangle_x = \{\tau\}$. If $\mathcal{M}, \tau, y, \vec{v}_{[x/v_1][y/v_2]} \vDash \varphi$, then $\mathcal{M}, \tau, x, \vec{v}_{[x/v_1][y/v_2]} \vDash \boxdot\downarrow^2_{\mathrm{T}} \varphi$ by **D31** and **T13**. If $\mathcal{M}, \tau, y, \vec{v}_{[x/v_1][y/v_2]} \nvDash \varphi$, then $\mathcal{M}, \tau, x, \vec{v}_{[x/v_1][y/v_2]} \vDash \boxdot\downarrow^2_{\mathrm{T}} \neg\varphi$ by the same reasoning. Thus $\mathcal{M}, \tau, x, \vec{v}_{[x/v_1][y/v_2]} \vDash \boxdot{\downarrow}^2_{\mathrm{T}}\neg\varphi \vee \boxdot{\downarrow}^2_{\mathrm{T}}\varphi$ by **D24**. Given the biconditionals $(*)$, we conclude this direction of the proof.

By the task frame $\mathcal{F}'$ and countermodel presented in **T13**, we may observe that $\sigma(0) = w_0 = \tau(0)$ where $\sigma, \tau \in \langle\tau\rangle_0$. Since $\tau(d) = w_0 \in |p|$ while $\sigma(d) = w_1 \notin |p|$, the clauses of **D31** give $\mathcal{M}, \tau, 0, \vec{v}_{[0/v_1][d/v_2]} \vDash {\downarrow}^2_{\mathrm{T}}p$ and $\mathcal{M}, \sigma, 0, \vec{v}_{[0/v_1][d/v_2]} \nvDash {\downarrow}^2_{\mathrm{T}}p$, and dually $\mathcal{M}, \tau, 0, \vec{v}_{[0/v_1][d/v_2]} \nvDash {\downarrow}^2_{\mathrm{T}}\neg p$ and $\mathcal{M}, \sigma, 0, \vec{v}_{[0/v_1][d/v_2]} \vDash {\downarrow}^2_{\mathrm{T}}\neg p$. Since $\tau, \sigma \in \langle\tau\rangle_0$, the world σ witnesses $\mathcal{M}, \tau, 0, \vec{v}_{[0/v_1][d/v_2]} \nvDash \boxdot{\downarrow}^2_{\mathrm{T}}p$ and τ itself witnesses $\mathcal{M}, \tau, 0, \vec{v}_{[0/v_1][d/v_2]} \nvDash \boxdot{\downarrow}^2_{\mathrm{T}}\neg p$. Thus $\mathcal{M}, \tau, 0, \vec{v}_{[0/v_1][d/v_2]} \nvDash \boxdot{\downarrow}^2_{\mathrm{T}}\neg p \vee \boxdot{\downarrow}^2_{\mathrm{T}}p$ where $d > 0$, and so the sentence fails at τ and 0 by $(*)$ above. $\square$

5.4 Soundness and Completeness

The modal axioms are sound by standard reasoning since $\Box$ quantifies universally over all possible worlds $H_{\mathcal{F}}$ without accessibility restriction, validating an S5 modal logic. Similarly, the temporal axioms are sound given that $\mathcal{D}$ is a nontrivial totally ordered abelian group, and so unbounded in both directions. The soundness proof for **TM** and its extensions has been formalized in the Lean 4 repository for this paper. I will present validity proofs for just the bimodal axiom **MF** and the reflection rule **TR**.

D34 The **S5** *Modal Logic* extends *Classical Propositional Logic* **CPL** to include all instances of the axiom schemata **MK**, **MT**, and **M5**, the rule **MP**, and the metarule **MN** presented in **§3.2**.

D35 The *Base Burgess–Xu Tense Logic* **BX** extends **CPL** to include the metarules **TN** and **TR** together with all instances of the axiom schemata **TS**, **TL**, **TC**, **UE**, **UT**, **UI**, **UC**, **UF**, **UG**, **SU**, **CN**, **NP**, **NF**, **NA**, and **NB** presented in **§3.2**.[90]

D36 The *Discrete Burgess–Xu Tense Logic* $\mathbf{BX}_{\mathrm{Z}}$ extends the base logic **BX** to include all instances of **UZ** and **Z1** presented in **§3.3**. Neither **UZ** nor **Z1** is due to Burgess or Xu; both are the present system's own principles for capturing $\mathbb{Z}$-time. Since **UZ** and **Z1** fail over every discrete temporal order that is not Archimedean (**P3**), and the Archimedean discrete orders are exactly $\mathbb{Z}$-time (**§3.3**), the discrete task frames over which $\mathbf{BX}_{\mathrm{Z}}$ and $\mathbf{TM}_{\mathrm{Z}}$ are sound and complete are exactly those over $\mathbb{Z}$-time.

D37 The *Dense Burgess–Xu Tense Logic* $\mathbf{BX}_{\mathrm{D}}$ extends the base logic **BX** to include all instances of **DN** and **NN** presented in **§3.3**. Neither **DN** nor **NN** is due to Burgess or Xu; both are the present system's own principles for the dense extension.

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[90] **TN** is (half of) Burgess's necessitation rule TG [69, §1.3], and **TS** is his *No Last Element* variant [69, §1.6]. **UC**, **UG**, **SU**, **UF**, and **UI** are his axioms A1a, A2a, A3a, A5a, and A6a respectively [69, §1.3], their past-tense duals following by **TR**, his working mirror-image convention. **CN** is his axiom A7a, independently confirmed by Xu as defining linear frames [70, §3]. **UE** is not among Burgess's numbered axioms but follows immediately from **UC** taking $\psi = \top$. **TC**, **UT**, **NP**, **NF**, **NA**, and **NB** are not found in Burgess or Xu and are the present system's own additions. **TL** corresponds in spirit to the local-linearity axioms Xu gives for his frame class V_3 [70, §3], though the exact correspondence among his formulas is not pinpointed here.

D38 The *Dense and Complete Burgess–Xu Tense Logic* $\mathbf{BX}_{\mathrm{R}}$ extends the dense logic $\mathbf{BX}_{\mathrm{D}}$ to include **PU** and **SEP** presented in **§3.3**. The completeness axiom **CO** of **§3.3** is a *derived theorem* of $\mathbf{BX}_{\mathrm{R}}$ rather than a further axiom, using only **PU** and the axioms of **BX**, and so is not included among the axioms of $\mathbf{BX}_{\mathrm{R}}$.

D39 The *Base Logic of Tense and Modality* **TM** for $\mathcal{L}$ extends **S5** and the base logic **BX** to include all instances of the *bimodal interaction* axiom **MF** presented in **§3.2**. The discrete $\mathbf{TM}_{\mathrm{Z}}$, dense $\mathbf{TM}_{\mathrm{D}}$, and dense and complete $\mathbf{TM}_{\mathrm{R}}$ extensions of **TM** include the additional axioms that distinguish $\mathbf{BX}_{\mathrm{Z}}$, $\mathbf{BX}_{\mathrm{D}}$, and $\mathbf{BX}_{\mathrm{R}}$, respectively. The derivation relation $\vdash_\Lambda$ for the system Λ is the smallest relation closed under the axioms and rules for Λ, omitting subscripts when ambiguity does not threaten.

D40 For each system Λ of **D39**, let C_Λ be its intended class of task frames: all task frames for **TM**, the dense task frames for $\mathbf{TM}_{\mathrm{D}}$, $\mathbb{Z}$-time for $\mathbf{TM}_{\mathrm{Z}}$, and $\mathbb{R}$-time for $\mathbf{TM}_{\mathrm{R}}$. The system Λ is *sound* just in case for every sentence φ and set of sentences Γ of $\mathcal{L}$, $\Gamma \vDash_{\mathsf{C}_\Lambda} \varphi$ (**D28**) whenever $\Gamma \vdash_\Lambda \varphi$.

L21 *Let $n : D \to D$ be defined by $n(x) \coloneqq -x$, where $\mathcal{F}^- = \langle W, \mathcal{D}, \Rightarrow^- \rangle$ where $w \Rightarrow^-_x u \coloneqq w \Rightarrow_{-x} u$, and $\tau^- = \tau \circ n$ for each $\tau \in H_{\mathcal{F}}$. Then $\mathcal{F}^-$ is a task frame and $\tau \mapsto \tau^-$ is a bijection $H_{\mathcal{F}} \to H_{\mathcal{F}^-}$, and for all well-formed sentences φ of $\mathcal{L}$, models $\mathcal{M} = \langle W, \mathcal{D}, \Rightarrow, |\cdot| \rangle$ over $\mathcal{F}$ and $\mathcal{M}^- = \langle W, \mathcal{D}, \Rightarrow^-, |\cdot| \rangle$ over $\mathcal{F}^-$, possible worlds $\tau \in H_{\mathcal{F}}$, and times $x \in D$, we have: $\mathcal{M}, \tau, x \vDash \varphi \;\Leftrightarrow\; \mathcal{M}^-, \tau^-, n(x) \vDash \varphi_{\langle \mathrm{S}|\mathrm{U} \rangle}$.*

Proof. Since $\mathcal{D}$ is a totally ordered abelian group, it follows that n is an order-reversing automorphism, where we have the following biconditionals:

$$x < y \;\Leftrightarrow\; -y < -x \;\Leftrightarrow\; n(y) < n(x) \tag{$\dagger$}$$

On the positive cone, $\Rightarrow^-$ is the converse of $\Rightarrow$, and so the composing direction of *Compositionality* on D^+ for $\Rightarrow^-$ is the converse of the composing direction for $\Rightarrow$, obtained by composing in reverse order, and so $v \Rightarrow_{y+x} w$ and hence $w \Rightarrow^-_{x+y} v$ since $y + x = x + y$. The factoring direction transfers by the mirror: given $w \Rightarrow^-_{x+y} v$ with $x, y \geqslant 0$ so that $v \Rightarrow_{x+y} w$, the factoring direction for $\Rightarrow$ with $x + y = y + x$ yields some $u \in W$ where $v \Rightarrow_y u$ and $u \Rightarrow_x w$, and so $w \Rightarrow^-_x u$ and $u \Rightarrow^-_y v$.

Seriality is immediate since the forward fibers of $\Rightarrow^-$ are the backward fibers of $\Rightarrow$ and conversely, so all are nonempty. *Limit* also transfers since the cones of $\Rightarrow^-$ coincide with those of $\Rightarrow$, and *Saturation* transfers since the fibers and segments coincide.

Since n is a bijection of D, the function $\tau^- = \tau \circ n$ is total. Given $a, b \in D$, we have $\tau^-(a) \Rightarrow^-_{b-a} \tau^-(b)$ just in case $\tau(-a) \Rightarrow_{a-b} \tau(-b)$, which holds by **D21** applied to τ at $-a, -b \in D$, and so $\tau^- \in H_{\mathcal{F}^-}$. Since $n \circ n = \mathrm{id}$ so that $(\mathcal{F}^-)^- = \mathcal{F}$, the same argument applied to $\mathcal{F}^-$ shows that $\sigma \in H_{\mathcal{F}^-}$ implies $\sigma^- \in H_{\mathcal{F}}$, and so $\tau \mapsto \tau^-$ is a bijection $H_{\mathcal{F}} \to H_{\mathcal{F}^-}$ with itself as inverse. The proof of the displayed biconditional is by induction on the complexity of φ.

Base Case (p_i): Since $(p_i)_{\langle \text{S}|\text{U}\rangle} = p_i$ and $\mathcal{M}, \mathcal{M}^-$ share the interpretation $|\cdot|$, the biconditional follows from **D24**:

$$\begin{aligned}
\mathcal{M}, \tau, x \vDash p_i \;&\Leftrightarrow\; \tau(x) \in |p_i| \\
&\Leftrightarrow\; \tau^-(n(x)) \in |p_i| \\
&\Leftrightarrow\; \mathcal{M}^-, \tau^-, n(x) \vDash p_i
\end{aligned}$$

The second biconditional holds since $\tau^-(n(x)) = \tau(n(n(x))) = \tau(x)$.

Base Case ($\bot$) *and Inductive Case* ($\varphi \to \psi$): Both are immediate from **D24**, the latter with the inductive hypothesis, since $\bot$ fails at every point and $(\varphi \to \psi)_{\langle \text{S}|\text{U}\rangle} = \varphi_{\langle \text{S}|\text{U}\rangle} \to \psi_{\langle \text{S}|\text{U}\rangle}$.

Inductive Case ($\Box\varphi$): Since $(\Box\varphi)_{\langle \text{S}|\text{U}\rangle} = \Box\varphi_{\langle \text{S}|\text{U}\rangle}$, the biconditional follows by **D24**:

$$\begin{aligned}
\mathcal{M}, \tau, x \vDash \Box\varphi \;&\Leftrightarrow\; \mathcal{M}, \sigma, x \vDash \varphi \text{ for all } \sigma \in H_{\mathcal{F}} \\
&\Leftrightarrow\; \mathcal{M}^-, \sigma^-, n(x) \vDash \varphi_{\langle \text{S}|\text{U}\rangle} \text{ for all } \sigma \in H_{\mathcal{F}} \\
&\Leftrightarrow\; \mathcal{M}^-, \rho, n(x) \vDash \varphi_{\langle \text{S}|\text{U}\rangle} \text{ for all } \rho \in H_{\mathcal{F}^-} \\
&\Leftrightarrow\; \mathcal{M}^-, \tau^-, n(x) \vDash \Box\varphi_{\langle \text{S}|\text{U}\rangle}
\end{aligned}$$

The second biconditional holds by the inductive hypothesis, the third since $\sigma \mapsto \sigma^-$ is a bijection $H_{\mathcal{F}} \to H_{\mathcal{F}^-}$, and the fourth by **D24** applied to $\mathcal{M}^-$ over $\mathcal{F}^-$.

Inductive Case ($\varphi \rhd \psi$): Writing φ' and ψ' for $\varphi_{\langle \text{S}|\text{U}\rangle}$ and $\psi_{\langle \text{S}|\text{U}\rangle}$, we may observe that $(\varphi \rhd \psi)_{\langle \text{S}|\text{U}\rangle} = \varphi' \lhd \psi'$, and so the biconditional follows by **D24**:

$$\begin{aligned}
\mathcal{M}, \tau, x \vDash \varphi \rhd \psi \;&\Leftrightarrow\; \mathcal{M}, \tau, z \vDash \psi \text{ for some } z > x \text{ where } \mathcal{M}, \tau, y \vDash \varphi \text{ for all } y \in (x, z) \\
&\Leftrightarrow\; \mathcal{M}^-, \tau^-, n(z) \vDash \psi' \text{ for some } z > x \\
&\;\; \text{where } \mathcal{M}^-, \tau^-, n(y) \vDash \varphi' \text{ for all } y \in (x, z) \\
&\Leftrightarrow\; \mathcal{M}^-, \tau^-, z' \vDash \psi' \text{ for some } z' < n(x) \\
&\;\; \text{where } \mathcal{M}^-, \tau^-, y' \vDash \varphi' \text{ for all } y' \in (z', n(x)) \\
&\Leftrightarrow\; \mathcal{M}^-, \tau^-, n(x) \vDash \varphi' \lhd \psi'
\end{aligned}$$

The second biconditional holds by the inductive hypothesis for φ and ψ. The third holds by (†), substituting $z' = n(z)$ and $y' = n(y)$, which reverses each inequality.

Inductive Case ($\varphi \lhd \psi$): By parity of reasoning with the $\rhd$ case, replacing $z > x$ with $z < x$ and (x, z) with (z, x), and applying (†) in the opposite direction. $\square$

T16 *If* $\vDash \varphi$, *then* $\vDash \varphi_{\langle \text{S}|\text{U}\rangle}$.

Proof. Suppose $\vDash \varphi$ and let $\mathcal{M} = \langle W, \mathcal{D}, \Rightarrow, |\cdot|\rangle$ be a model defined over the task frame $\mathcal{F} = \langle W, \mathcal{D}, \Rightarrow\rangle$, where $\sigma \in H_{\mathcal{F}}$ and $y \in D$ be arbitrary. Let $\mathcal{F}^- = \langle W, \mathcal{D}, \Rightarrow^-\rangle$ and $\mathcal{M}^- = \langle W, \mathcal{D}, \Rightarrow^-, |\cdot|\rangle$ be as in **L21**. Since $\mathcal{F}^-$ is itself a task frame, $\mathcal{M}^-$ is a model, and since $\vDash \varphi$ holds over every model, we have $\mathcal{M}^-, \sigma^-, n(y) \vDash \varphi$, where $\sigma^- \in H_{\mathcal{F}^-}$ by the bijection established in **L21**.

Applying **L21** to $\mathcal{M}^-$ over $\mathcal{F}^-$ with $\tau = \sigma^-$ and $x = n(y)$ yields:

$$(\mathcal{M}^-)^-, (\sigma^-)^-, n(n(y)) \vDash \varphi_{\langle \text{S}|\text{U} \rangle}.$$

Since $(\mathcal{F}^-)^- = \mathcal{F}$ so that $(\mathcal{M}^-)^- = \mathcal{M}$, and since $(\sigma^-)^- = \sigma$ and $n(n(y)) = y$, we conclude $\mathcal{M}, \sigma, y \vDash \varphi_{\langle \text{S}|\text{U} \rangle}$ as desired. Equivalently, $\vDash \varphi_{\langle \text{S}|\text{U} \rangle}$. $\square$

T17 $\quad \vDash \Box\varphi \rightarrow \Box\boxed{\text{F}}\varphi$.

Proof. Suppose $\mathcal{M}, \tau, x \vDash \Box\varphi$. By **D24**, $\mathcal{M}, \sigma, x \vDash \varphi$ for all $\sigma \in H_{\mathcal{F}}$. Let $\sigma \in H_{\mathcal{F}}$ and $y > x$ be arbitrary. By **L17**, there is some $\rho \in H_{\mathcal{F}}$ where $\sigma \approx_y^x \rho$, and $\mathcal{M}, \rho, x \vDash \varphi$ by **D24** since $\mathcal{M}, \tau, x \vDash \Box\varphi$. Thus $\mathcal{M}, \sigma, y \vDash \varphi$ by **L18**. Since $y > x$ and σ were arbitrary, we may conclude that $\vDash \Box\varphi \rightarrow \Box\boxed{\text{F}}\varphi$ as desired. $\square$

T18 (Soundness) $\quad$ *If* $\vdash \varphi$*, then* $\vDash \varphi$.

Proof. By induction on derivations. The propositional tautologies are valid, and **MP** preserves validity by standard arguments. The S5 axioms **MK**, **MT**, and **M5** are valid given that $\Box$ quantifies universally over $H_{\mathcal{F}}$ and **MN** preserves validity.

The temporal axioms are valid given that $\mathcal{D}$ is a nontrivial totally ordered abelian group, where **TN** preserves validity since a sentence valid at every point is valid at every later point, and **TR** preserves validity since there is an order-reversing automorphism $x \mapsto -x$ on $\mathcal{D}$.

The bimodal axiom **MF** is shown to be valid in **T17**. The proof of soundness otherwise proceeds as usual. The Lean 4 repository for this paper formally verifies this soundness result for **TM** and its extensions. $\square$

C5 $\quad$ *The perpetuity principles* ***P1*** *–* ***P6*** *are all valid.*

Proof. Follows immediately from **T18** given the derivations of **P1** – **P6** in §**3.2**. $\square$

C6 (Completeness) $\quad$ *A proof system* **S** *is strongly complete over a class* C *of task frames just in case* $\Gamma \vDash_{\mathsf{C}} \varphi$ *(****D28****) implies* $\Gamma \vdash_{\mathbf{S}} \varphi$ *for every set of sentences* Γ*, and weakly complete over* C *just in case* $\vDash_{\mathsf{C}} \varphi$ *implies* $\vdash_{\mathbf{S}} \varphi$*. Completeness is then carried by the following* $\mathcal{L}$ *systems:*

TM $\quad$ *Strongly complete over all task frames.*

$\mathbf{TM_d}$ $\quad$ *Strongly complete over the dense task frames.*

$\mathbf{TM_z}$ $\quad$ *Weakly complete over* $\mathbb{Z}$*-time.*

$\mathbf{TM_r}$ $\quad$ *Weakly complete over* $\mathbb{R}$*-time.*

Strong completeness provably fails for $\mathbb{Z}$*-time as well as* $\mathbb{R}$*-time where compactness fails, and so weak completeness is the appropriate target.*[91]

[91] These results, together with the soundness of the corresponding systems, are established in the Lean 4 repository for this paper, and so their proofs are not reproduced here. Each system is also extended to

Proof. Verified in the Lean 4 repository for this paper. □

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