The Construction of Possible Worlds

Tense and modal reasoning about non-deterministic systems

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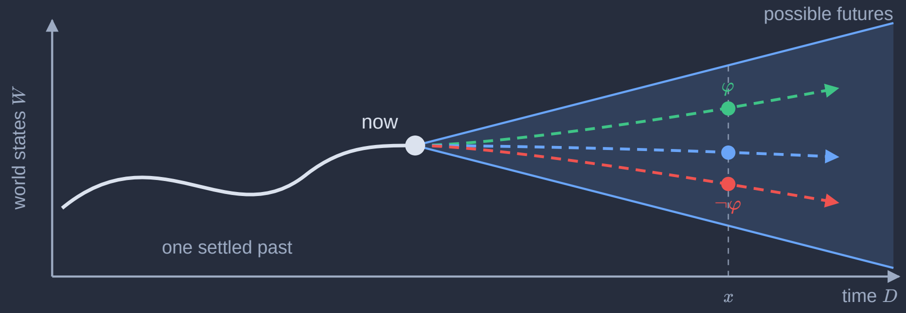
Lean 4 formalization: `github.com/benbrastmckie/BimodalLogic`

The Open Future

- **Future contingency** is characteristic of **non-deterministic dynamical systems**
- Each state does not determine a unique future, leaving many futures open

many possible continuations



one past, many possible futures

Both **tense** and **modal** operators are needed to reason about non-deterministic systems

- **Tense Operators** — for reasoning about past and future times within a history
- **Historical Modal Operators** — for reasoning about different histories

Tense — Reasoning within a world history

- **Temporary Sentences**: "Cary is reading" — true this morning, false this afternoon, *same world history*
- **Primitive Tense Operators** (Kamp 1968): $\varphi \lhd \psi$ "φ has been the case *since* ψ" $\varphi \rhd \psi$ "φ will be the case *until* ψ"

Everything else is defined:

Some Past $P\varphi := \top \lhd \varphi$ *Some Future* $F\varphi := \top \rhd \varphi$

All Past $H\varphi := \neg P \neg \varphi$ *All Future* $G\varphi := \neg F \neg \varphi$

Always $\triangle\varphi := H\varphi \wedge \varphi \wedge G\varphi$ *Sometimes* $\triangledown\varphi := P\varphi \vee \varphi \vee F\varphi$

Next $\text{Ⓕ}\varphi := \bot \rhd \varphi$ *Previous* $\text{Ⓟ}\varphi := \bot \lhd \varphi$

> **Note:** Ⓕ and Ⓟ are only worth having for *discrete* systems — chess moves, clock cycles, etc. — and express $\bot$ at points with no successor/predecessor

Tense operators move the point of evaluation **along one world history** — either earlier or later

- Although Cary is reading, she **could** have spent the morning sleeping
- But to say what **could** have happened, we must reason about **other world histories**, not just other times

Historical Modality — Reasoning about world histories

- **Modal Operators:** Quantify over *all/some* world histories
  - $\Box\varphi$ reads "it is necessary that φ" — φ holds in **every** world history
  - $\Diamond\varphi := \neg\Box\neg\varphi$ reads "it is possible that φ" — φ holds in **some** world history
- **Example:** Taking φ to be the sentence "Cary is reading":
  - $\Diamond\varphi$ is true since some world history has her reading
  - $\Box\varphi$ is false since she is not reading in every world history
- Neither tense nor modality says anything about the other

Perpetuity Principles — constrain the interaction between tense and modality:

- **P1**: $\Box\varphi \rightarrow \triangle\varphi$ — "what is necessary is always the case."
- **P2**: $\triangledown\varphi \rightarrow \Diamond\varphi$ — "what is sometimes the case is possible."

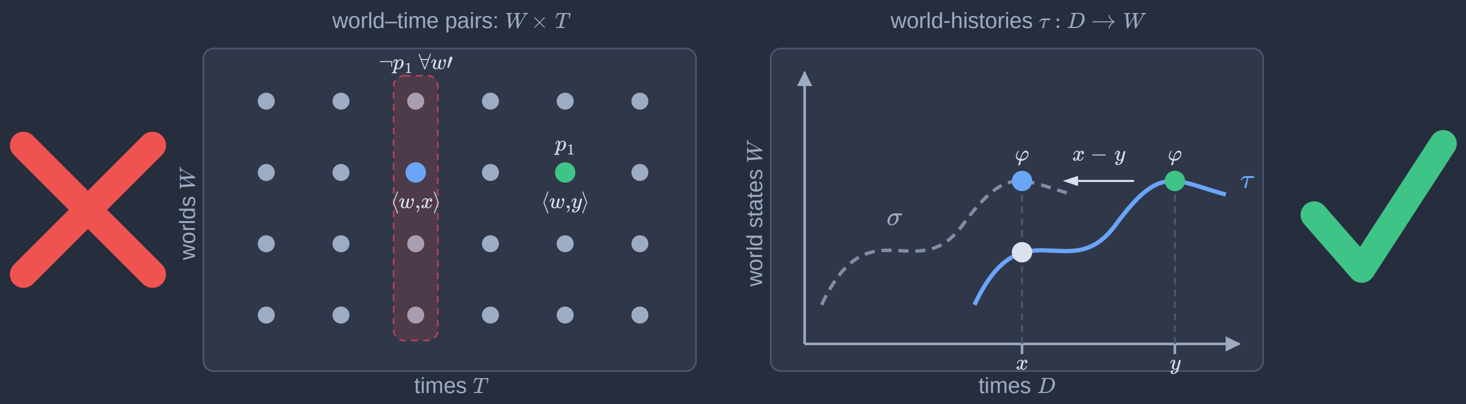
- These **schematic principles** ought to hold for all substitutions of φ
- **P1** and **P2** are interderivable (substituting $\neg\varphi$ and contraposing)

Bimodal Reasoning

- **Modal logic** and **tense logic** — both well understood individually
- **Tense × Modality:** Naive product invalidates perpetuity: $\triangledown\varphi \to \Diamond\varphi$ — what is sometimes the case is possible.

Putting logics together is not as simple as taking the cross-product.

- A natural alternative comes from **dynamical systems theory** used throughout physics
- The space of world histories for a system are modeled by **functions from times to world states**



Roadmap — Four Positions

1 · Kripke 1959, 1963

- Set of **possible worlds** W and **relation** $R \subseteq W \times W$
- W might represent *total instantaneous configurations*
- Rwu might represent *transitions* between $w, u \in W$

2 · Prior 1957, 1967

- Provides an **time order** $<$ on the **world-states** T
- Also refers to $x \in T$ as *times* as well as *moments*
- Requires $<$ to be a *connected left linear partial order*

3 · Montague / Kaplan 1973, 1989

- Both possible worlds W and times T are primitive
- Evaluate sentences at world-time pairs $\langle w, x \rangle$
- Present semantic theories for bimodal languages

4 · Task Semantics: $w, u \in W$ and $x, y \in D$

- Possible transition $w \Rightarrow_x u$ from w to u in x duration
- World histories are functions from times to world-states
- Constrained by $\tau(x) \Rightarrow_{y-x} \tau(y)$ for all $x, y \in D$

Kripke: Went on the provide a theory of *metaphysical modality* in "Naming and Necessity" (1972)

Prior: Concerned with tense logics that admit of an open future

Montague / Kaplan: Concerned with natural language semantics for both tense and modality

Present Concern: Logical reasoning about *non-deterministic dynamical systems*

1. Kripke Semantics

Frame $\mathcal{F} = \langle W, R \rangle$ — accessibility $R \subseteq W^2$ between worlds in $W \neq \varnothing$

Model $\mathcal{M} = \langle \mathcal{F}, |\cdot| \rangle$, with $|p_i| \subseteq W$ for all **sentences letters** $p_1, p_2, \ldots$

Satisfaction: Truth is defined recursively for all sentences:

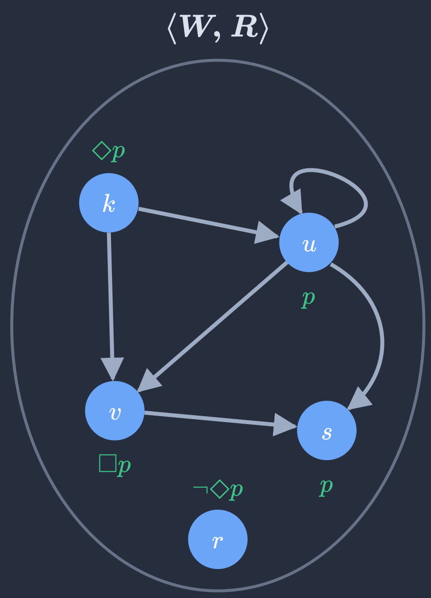
$\mathcal{M}, w \nvDash \bot$	$\neg\varphi := \varphi \to \bot$
$\mathcal{M}, w \vDash p_i$ iff $w \in \lvert p_i \rvert$	$\varphi \vee \psi := \neg\varphi \to \psi$
$\mathcal{M}, w \vDash \varphi \to \psi$ iff $\mathcal{M}, w \nvDash \varphi$ or $\mathcal{M}, w \vDash \psi$	$\varphi \wedge \psi := \neg(\varphi \to \neg\psi)$
$\mathcal{M}, w \vDash \Box\varphi$ iff $\mathcal{M}, u \vDash \varphi$ for all $u \in W$ with wRu	

Consequence: $\Gamma \vDash \varphi$ iff $\mathcal{M}, w \vDash \varphi$ whenever $\mathcal{M}, w \vDash \psi$ for all $\psi \in \Gamma$

Frame Validity: $\vDash_{\mathcal{F}} \varphi$ iff $\mathcal{M}, w \vDash \varphi$ for every $\mathcal{M}$ on $\mathcal{F}$ and every $w \in W$

Correspondence: the axioms are valid on the following frames:

T	$\Box\varphi \to \varphi$	reflexive: wRw
B	$\varphi \to \Box\Diamond\varphi$	symmetric: $wRu \Rightarrow uRw$
4	$\Box\varphi \to \Box\Box\varphi$	transitive: $wRu \,\&\, uRv \Rightarrow wRv$
D	$\Box\varphi \to \Diamond\varphi$	serial: wRu for some u
5	$\Diamond\varphi \to \Box\Diamond\varphi$	euclidean: $wRu \,\&\, wRv \Rightarrow uRv$



no temporal structure

Model on this frame: $|p| = \{u, s\}$

$\mathcal{M}, k \vDash \Diamond p$ iff k "sees" some p-world

- Could be read: it **must** be the case that p

$\mathcal{M}, k \vDash \Box p$ iff every world that k "sees" is a p-world

- Could be read: it **might** be the case that p

2. Prior’s Linear Semantics

Frame $\mathcal{T} = \langle T, < \rangle$ where $<$ is a **strict total order** on the **times** in $T \neq \varnothing$

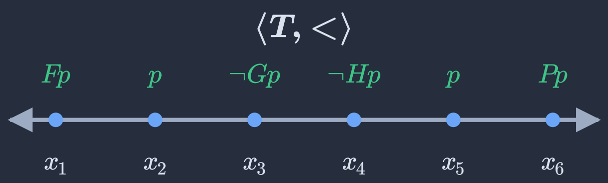
Model $\mathcal{P} = \langle \mathcal{T}, |\cdot| \rangle$, with $|p_i| \subseteq T$ for all **sentence letters** $p_1, p_2, \ldots$

Satisfaction at time x is defined recursively for all sentences:

$\mathcal{P}, x \vDash p$ iff $x \in |p|$ $\qquad$ $\mathcal{P}, x \nvDash \bot$

$\mathcal{P}, x \vDash \varphi \to \psi$ iff $\mathcal{P}, x \nvDash \varphi$ or $\mathcal{P}, x \vDash \psi$

$\mathcal{P}, x \vDash H\varphi$ iff $\mathcal{P}, y \vDash \varphi$ for all $y < x$ $\qquad$ $\mathcal{P}, x \vDash G\varphi$ iff $\mathcal{P}, y \vDash \varphi$ for all $y > x$



Model on this frame: $|p| = \{x_2, x_5\}$

$\mathcal{P}, x \vDash Pp$ iff some $y < x$ is in $|p|$

$\mathcal{P}, x \vDash Fp$ iff some $y > x$ is in $|p|$

Assumes only **one future** is possible

Correspondence: the axioms below are valid on exactly the frames whose $<$ meets the corresponding conditions:

T4	$G\varphi \to GG\varphi$	transitive: $x<y \;\&\; y<z \Rightarrow x<z$
TS	$F\top$	serial: $x < y$ for some y
TL	$(P\varphi \wedge P\psi) \to \big[P(P\varphi \wedge \psi) \vee P(\varphi \wedge \psi) \vee P(\varphi \wedge P\psi)\big]$	left-linear: any two pasts comparable
DF	$(H\varphi \wedge \varphi \wedge F\top) \to FH\varphi$	discrete: least later time
DN	$GG\varphi \to G\varphi$	dense: $\exists z,\; x<z<y$
CO	$\triangle(H\varphi \to FH\varphi) \to (H\varphi \to G\varphi)$	complete: bounded sets have a lub

2. Prior's Ockhamist Semantics

Frame $\mathcal{T} = \langle T, < \rangle$ is a **connected left-linear strict partial order**

A **strict history** $h = \langle T_h, <_h \rangle$ is any **maximal total suborder** of $\mathcal{T}$

$H_{\mathcal{T}}$ is the set of all strict histories of $\mathcal{T}$ and $H^x_{\mathcal{T}} := \{h \in H_{\mathcal{T}} : x \in T_h\}$

Model $\mathcal{P} = \langle \mathcal{T}, |\cdot| \rangle$, with $|p_i| \subseteq T$ for all **sentence letters** $p_1, p_2, \ldots$

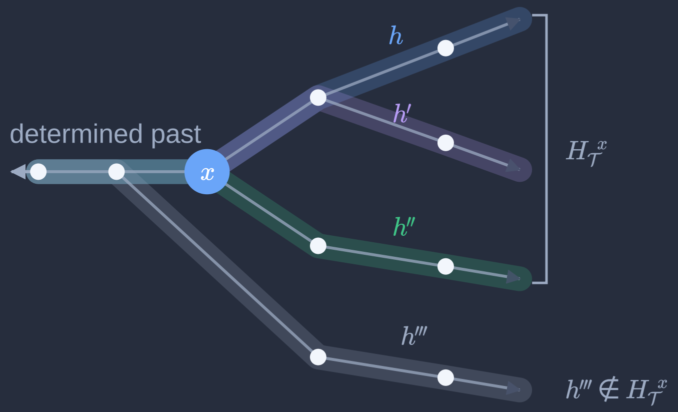
Satisfaction in history h at time x is defined recursively:

$\mathcal{P}, h, x \vDash p_i$ iff $x \in h$ and $x \in |p_i|$ (Booleans are similar to before)

$\mathcal{P}, h, x \vDash H\varphi$ iff $\mathcal{P}, h, y \vDash \varphi$ for all $y \in h$ with $y < x$

$\mathcal{P}, h, x \vDash G\varphi$ iff $\mathcal{P}, h, y \vDash \varphi$ for all $y \in h$ with $x < y$

$\mathcal{P}, h, x \vDash \boxdot\varphi$ iff $\mathcal{P}, h', x \vDash \varphi$ for all $h' \in H^x_{\mathcal{T}}$ (**stability** modal)

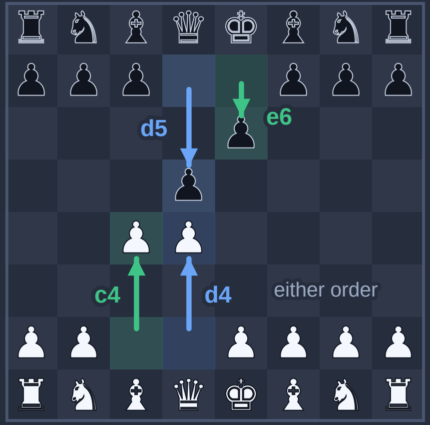


Each circled path is a **strict history**

Limitations:

- No recurrence of the same configuration
- Can't distinguish **what** state a system is in from **when** it is in it
- No transpositions in the order of configuration states
- No quantification over histories that do not intersect the present

recurrence



transposition

3. Two-Dimensional Semantics

Frame $\mathcal{F}_2 = \langle W, T, < \rangle$ — **primitive** worlds W, times T; **strict total order** $<$

Model $\mathcal{M}_2 = \langle \mathcal{F}_2, |\cdot| \rangle$, with $|p_i| \subseteq W \times T$ for all **sentence letters** $p_1, p_2, \ldots$

Satisfaction in world w at time x is defined recursively:

$\mathcal{M}_2, w, x \vDash p_i$ iff $\langle w, x \rangle \in |p_i|$ (Booleans are similar to before)

$\mathcal{M}_2, w, x \vDash H\varphi$ iff $\mathcal{M}_2, w, y \vDash \varphi$ for all $y \in T$ with $y < x$

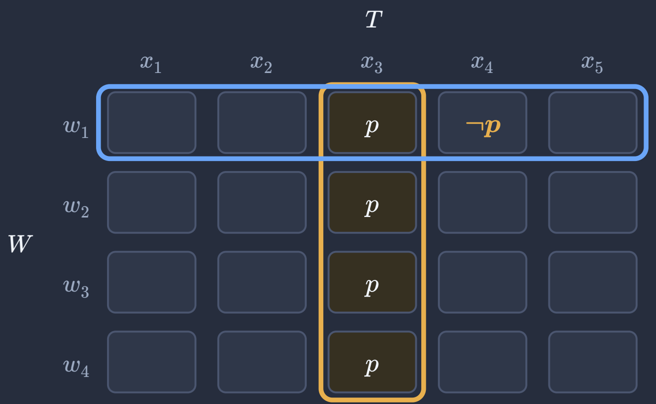
$\mathcal{M}_2, w, x \vDash G\varphi$ iff $\mathcal{M}_2, w, y \vDash \varphi$ for all $y \in T$ with $x < y$

$\mathcal{M}_2, w, x \vDash \Box\varphi$ iff $\mathcal{M}_2, u, x \vDash \varphi$ for all $u \in W$ (**Kaplan**: time fixed)

$\mathcal{M}_2, w, x \vDash \boxtimes\varphi$ iff $\mathcal{M}_2, u, y \vDash \varphi$ for all $u \in W$ and $y \in T$ (**Montague**: all times)

Whereas $\Box$ scans a **column**, $\triangle$ scans a **row** — nothing links the two

- Kaplan's $\Box$ makes **P1** $\Box\varphi \to \triangle\varphi$ **invalid**
- Montague's $\boxtimes$ ("*necessarily always*") validates $\boxtimes\varphi \to \triangle\varphi$ **trivially**
  - $\vDash \boxtimes\varphi \leftrightarrow \Box\triangle\varphi$ is valid and so $\boxtimes\varphi := \Box\triangle\varphi$ may be defined
  - So $\boxtimes\varphi \to \triangle\varphi$ is an instance of **T**
  - $\Box$ is **not definable** from $\boxtimes$, H, G



$\boxtimes p$ at $\langle w_1, x_3 \rangle$: p in every world w at every time x

$\Box p$ at $\langle w_1, x_3 \rangle$: p in every world at x_3

$\triangle p$ at $\langle w_1, x_3 \rangle$: p at every time in w_1 — fails at x_4

Model on this frame: $|p| = W \times \{x_3\}$ — p true in the yellow column only

$\mathcal{M}_2, w_1, x_3 \vDash \Box p$ since p holds in **every world** at x_3

$\mathcal{M}_2, w_1, x_3 \nvDash \triangle p$ since p fails in w_1 at x_4 — already $\nvDash Gp$

P1 $\Box p \to \triangle p$ fails at $\langle w_1, x_3 \rangle$

However, what is necessary is surely always the case

> *"Physics should have helped us to realize that a temporal theory of a phenomenon X is, in general, more than a simple combination of two components: the statics of X and the ordered set of temporal instants. … The general moral, then, is that we shouldn't expect the theory of time + X to be obtained by mechanically combining the theory of time and the theory of X."*
>
> — Richmond Thomason (1984, p. 135)

Since Galileo, Newton, and Leibniz, the sciences have represented the evolution of a system with **functions**.

4. Task Semantics

A **task frame** $\mathcal{F} = \langle W, \mathcal{D}, \Rightarrow \rangle$ is a *labeled transition system* with:

- **World states** $W \neq \varnothing$ represent total instantaneous configurations
- **Durations** $\mathcal{D} = \langle D, +, 0, \leq \rangle$ forming a nontrivial ordered abelian group
- A **task relation** $w \Rightarrow_x u$ with $x \geq 0$ for a *transition* from w to u in duration x
- **Negative durations** are defined by the converse $w \Rightarrow_{-x} u := u \Rightarrow_x w$

A **partial history** is a function $\tau : X \to W$ where $\tau(x) \Rightarrow_{y-x} \tau(y)$ for all $x, y \in X$

- A **world history** is any partial history that includes all durations $X = D$
- $H_{\mathcal{F}}$ is the **set of all world histories** for the task frame $\mathcal{F}$

Model $\mathcal{M} = \langle \mathcal{F}, |\cdot| \rangle$ where $|p_i| \subseteq W$ for each sentence letter $p_1, p_2, \ldots$

Language $\varphi, \psi ::= p_i \mid \bot \mid \varphi \to \psi \mid \Box\varphi \mid \varphi \lhd \psi \mid \varphi \rhd \psi$

Satisfaction at a possible world $\tau \in H_{\mathcal{F}}$ and time $x \in D$:

$\mathcal{M}, \tau, x \vDash p_i$ iff $\tau(x) \in |p_i|$ (Booleans as before)

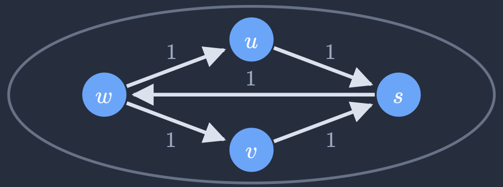
$\mathcal{M}, \tau, x \vDash \varphi \lhd \psi$ iff $\mathcal{M}, \tau, z \vDash \psi$ for some $z < x$ with $\mathcal{M}, \tau, y \vDash \varphi$ for all $z < y < x$

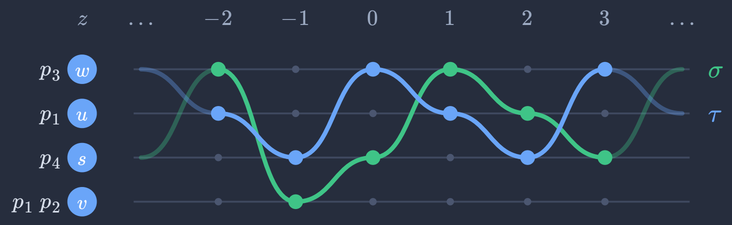
$\mathcal{M}, \tau, x \vDash \varphi \rhd \psi$ iff $\mathcal{M}, \tau, z \vDash \psi$ for some $z > x$ with $\mathcal{M}, \tau, y \vDash \varphi$ for all $x < y < z$

$\mathcal{M}, \tau, x \vDash \Box\varphi$ iff $\mathcal{M}, \sigma, x \vDash \varphi$ for all $\sigma \in H_{\mathcal{F}}$

$\mathcal{M}, \tau, x \vDash \boxdot\varphi$ iff $\mathcal{M}, \sigma, x \vDash \varphi$ for all $\sigma \in H_{\mathcal{F}}$ where $\sigma(x) = \tau(x)$

$\langle W, \mathcal{D}, \Rightarrow \rangle$





Combined modals may be defined with $\Diamonddot\varphi := \neg\boxdot\neg\varphi$ as follows:

Will always $\boxed{!}\varphi := \boxdot G\varphi$ **Will eventually** $\Diamond_{!}\varphi := \boxdot F\varphi$

Could always $\boxed{?}\varphi := \Diamonddot G\varphi$ **Could eventually** $\Diamond_{?}\varphi := \Diamonddot F\varphi$

Invariants — properties of the *system*, in the vocabulary above:

- $\Box(p_2 \to p_1)$ — asserts that $|p_2| \subseteq |p_1|$
- $\Box\Diamond_{!} p_3$ — whatever the state, w lies ahead: **unavoidable**, and so **recurrent**
- $\Diamond p_2 \wedge \Diamond\triangle\neg p_2$ — v is possible, yet **avoidable**
- $\Box(\boxed{?}\neg p_2 \wedge \Diamond_{?} p_2)$ — no state settles whether v is ahead: **open future**
- $\Box(p_3 \to \Diamond_{!}(p_4 \wedge \Diamond_{!} p_3))$ — after any w comes s, then w again: the **cycle closes**

4. Task Frames

Officially, the task frames are must satisfy constraints:

Fiber $\mathrm{fib}(w,x) := \{u : w \Rightarrow_x u\}$ — worlds states w can get to in x duration

Cone $(w)_x := \bigcup_{|y|<x} \mathrm{fib}(w,y)$ — every world state within $(-x,x)$ around w

Segment $[w,v]_x^y := \mathrm{fib}(w,x) \cap \mathrm{fib}(v,-y)$ — from w after x *and* from v before y

Task frames must satisfy four constraints, for all $x, y \geq 0$:

Compositionality $w \Rightarrow_{x+y} v$ iff $w \Rightarrow_x u \Rightarrow_y v$ for some u — tasks compose

Seriality $w \Rightarrow_x u$ and $v \Rightarrow_x w$ for some u, v — no dead ends, in either direction

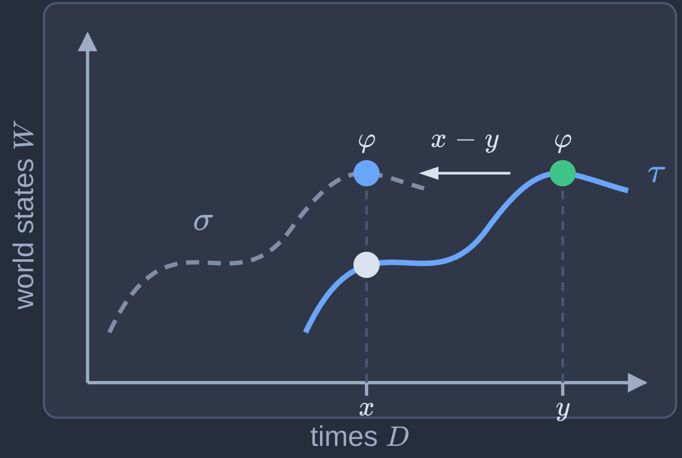
Limit $\bigcap_{x>0}(w)_x = \{w\}$ — distinct states are instantaneously separated

Saturation $\bigcap \mathcal{S} \neq \varnothing$ for $\supseteq$-directed $\mathcal{S}$ of nonempty fibers and segments

What the constraints buy:

- Every partial history extends to a possible world
- The value for each new time z is constrained by settled times
- Nearer assignments subsume farther ones — the constraints **nest**
- **Saturation** ensures candidates even at the limit
- Finite W makes every fiber/segment finite so Saturation is free

world-histories $\tau : D \to W$



Every possible world can be time-shifted:

- $\sigma(z) := \tau(z+d)$ is again a path for any $d \in D$
- The world-history condition $\tau(x) \Rightarrow_{y-x} \tau(y)$ sees only *differences* in times
- Interpretations $|p_i| \subseteq W$ do not include any times at all

Perpetuity is valid: $\vDash \triangledown\varphi \to \Diamond\varphi$

- Let $\mathcal{M}, \tau, x \vDash \triangledown\varphi$, so $\mathcal{M}, \tau, y \vDash \varphi$ for some $y \in D$
- Define $\sigma(z) := \tau(z+y-x)$ which is a possible world where $\sigma(x) = \tau(y)$
- Shifting preserves truth, by induction on φ: so $\mathcal{M}, \sigma, x \vDash \varphi$
- So $\sigma \in H_{\mathcal{F}}$ witnesses $\mathcal{M}, \tau, x \vDash \Diamond\varphi$, and so $\vDash \triangledown\varphi \to \Diamond\varphi$

The Logic TM

TM $= \mathbf{S5} + \mathbf{BX} + \mathbf{MF}$, defined for $\varphi, \psi ::= p_i \mid \bot \mid \varphi \to \psi \mid \Box\varphi \mid \varphi \lhd \psi \mid \varphi \rhd \psi$ where $\varphi_{\langle \mathrm{s}|\mathrm{u}\rangle}$ interchanges $\lhd$ with $\rhd$ throughout φ

Modal Logic S5

MP	$\varphi,\ \varphi \to \psi \vdash \psi$	modus ponens
MN	if $\vdash \varphi$ then $\vdash \Box\varphi$	necessitation
MK	$\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$	distribution
MT	$\Box\varphi \to \varphi$	reflexive
M5	$\Diamond\Box\varphi \to \Box\varphi$	euclidean

Tense Logic BX — core

TN	if $\vdash \varphi$ then $\vdash G\varphi$	temporal necessitation
TR	if $\vdash \varphi$ then $\vdash \varphi_{\langle \mathrm{s}|\mathrm{u}\rangle}$	time reflection
TS	$F\top$	serial
TC	$\varphi \to GP\varphi$	now is past to every future
UE	$(\varphi \rhd \psi) \to F\psi$	until implies eventually
UT	$F\varphi \to (\top \rhd \varphi)$	eventually implies until

Linearity and connectedness

TL	$(F\varphi \wedge F\psi) \to \big[F(F\varphi \wedge \psi) \vee F(\varphi \wedge \psi) \vee F(\varphi \wedge F\psi)\big]$	right-linear: any two futures comparable
CN	$\big[(\varphi \rhd \psi) \wedge (\chi \rhd \theta)\big] \to \big[(\varphi \wedge \chi) \rhd (\psi \wedge \theta) \vee (\varphi \wedge \chi) \rhd (\psi \wedge \chi) \vee (\varphi \wedge \chi) \rhd (\varphi \wedge \theta)\big]$	connected

Bimodal interaction

MF	$\Box\varphi \to \Box G\varphi$	what is necessary is necessarily always going to be the case, from which perpetuity may be derived

The Logic TM — Axioms and Completeness

Until axioms — Kamp's *since* and *until* as primitives

UI	$\varphi \rhd (\varphi \wedge (\varphi \rhd \psi)) \to \varphi \rhd \psi$	idempotence
UC	$G(\varphi \to \psi) \to ((\chi \rhd \varphi) \to (\chi \rhd \psi))$	consequent monotone
UF	$(\varphi \rhd \psi) \to (\varphi \wedge (\varphi \rhd \psi)) \rhd \psi$	the guard holds throughout
UG	$G(\varphi \to \chi) \to ((\varphi \rhd \psi) \to (\chi \rhd \psi))$	guard monotone
SU	$\theta \wedge (\varphi \rhd \psi) \to \varphi \rhd (\psi \wedge (\varphi \lhd \theta))$	since and until interlock

Uniformity — vacuous unless time is discrete

NP	$\textcircled{F}\top \to \textcircled{P}\top$	successors iff predecessors
NF	$\textcircled{F}\top \to G\textcircled{F}\top$	and so at every later time
NA	$\textcircled{F}\top \to H\textcircled{F}\top$	and at every earlier time
NB	$\textcircled{F}\top \to \Box\textcircled{F}\top$	and in every world history

BX$_z$ — Discrete extension

UZ	$F\varphi \to (\neg\varphi \rhd \varphi)$	a nearest future φ-time
Z1	$G(G\varphi \to \varphi) \to (FG\varphi \to G\varphi)$	backward induction

BX$_d$ — Dense extension

DN	$GG\varphi \to G\varphi$	dense
NN	$\neg\textcircled{F}\top$	no immediate successor

BX$_r$ — Dense and complete extension

Reynolds' operators $K^+\varphi := \neg(\neg\varphi \rhd \top)$ $K^-\varphi := \neg(\neg\varphi \lhd \top)$

PU	$(\varphi \rhd \top) \wedge F\neg\varphi \to \varphi \rhd (\neg\varphi \vee K^+\neg\varphi)$	no gap at the end
SEP	$K^+\varphi \wedge \neg K^+(\varphi \wedge (\neg\varphi \rhd \varphi)) \to K^+(K^+\varphi \wedge K^-\varphi)$	separation

Sound and **complete** — each extension carries over to **TM**, all formalized in **Lean 4**: github.com/benbrastmckie/BimodalLogic

TM	*strongly* complete over all task frames	**TM$_d$**	*strongly* complete over the dense frames
TM$_z$	*weakly* complete over $\mathbb{Z}$-time	**TM$_r$**	*weakly* complete over $\mathbb{R}$-time

The Logos — A Unified Logic

Task frames provide a **starting point**, not a final theory. The *Logos* extends the semantics to support a wide range of additional operators, adding the necessary parameters at which to evaluate sentences in the Logos.

Foundations — one hyperintensional task semantics

Tense & modal THIS TALK
$\triangleleft, \triangleright, H, G, P, F, ⓕ, ⓟ, \Box, \Diamond, \boxdot, \Diamond$
(Did not mention: $\downarrow, \uparrow$)

Counterfactual
$\varphi \mathrel{\Box\!\!\rightarrow} \psi$, $\varphi \mathrel{\Diamond\!\!\rightarrow} \psi$ — impose a φ state on a world, these claim ψ *would* or *might* have been, respectively

Causal
$\varphi \mathrel{\circ\!\!\rightarrow} \psi$ — explanation over a duration: what *brings about* what

Constitutive
propositional identity $\equiv$, ground $\leq$, and essence $\sqsubseteq$ for non-monotonic explanatory reasoning

Extensions — modular plugins, combinable in any subset

Epistemic
$\mathsf{K}, \mathsf{B}, \mathsf{Pr}, \mathsf{Cr}$, indicatives $\hookrightarrow$ — information as a *part* of the world; chance and credence over histories

Normative
$\mathsf{O}, \mathsf{P}, \preceq$, utility — obligation, permission, preference; value carried by histories

Agential
$\mathsf{Can}, \mathsf{Stit}, \mathsf{Ch}, \mathsf{Do}$ — ability, choice, actions, strategies, free permission, indexed to agents

Spatial
Dist, located @ — distance, location, extended bodies across time

> **A training signal for reasoning**: the logic and semantic theory give a verdict on every logical inference $\Gamma \vdash \varphi$ — proof terms in Lean 4 provide a **positive signal** and countermodels found by Z3 provide a **corrective signals**, showing *why* a valid inference holds or *how* an invalid inference fails. Soundness keeps the two from conflicting; the data is bounded by computation, not human annotation.

Thanks

PAPER

The Construction of Possible Worlds

Forthcoming in *J. Philosophical Logic*.

benbrastmckie.com/publications/possible_worlds.pdf

LEAN 4 FORMALIZATION

BimodalLogic

Soundness, Completeness, Decidability.

github.com/benbrastmckie/BimodalLogic

GRADUATE SEMINAR @ MIT

Modern History of Modal Logic

Readings and notes this talk retraces, from Kripke and Prior to the present.

github.com/benbrastmckie/ModalHistory

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